Section 1.3 – Measurement of Rotation – Day 1

Objective: Given the measure of an angle, be able to draw a picture of the angle.

An angle is used to measure an amount of rotation. We will first measure angle rotations in degrees, denoted with a degree symbol $\circ$.

What is the angle of rotation if you stand and do an about face (turn around and face backwards)?
Answer:

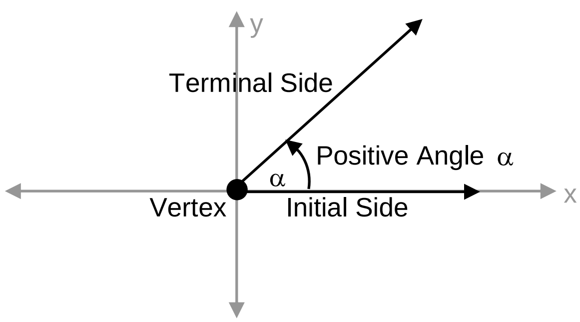
Greek letters are often used to denote angles. Some frequently used letters to denote angles include the following:

α :	alpha	γ :	gamma	ϕ :	phi
β :	beta	ω:	omega	θ :	theta

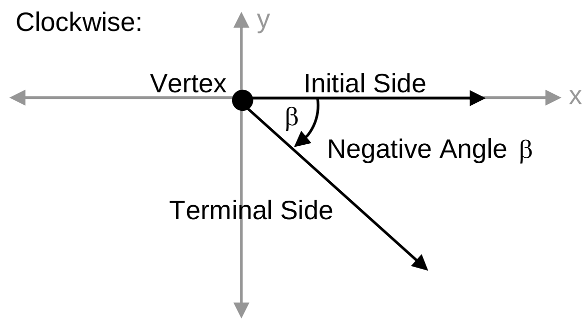
A ray, or half-line, is that portion of a line that starts at a point V on the line and extends infinitely in one direction. The starting point V of a ray is called its vertex.

Two rays that are drawn with a common vertex form an angle. One of the rays of an angle is called the initial side and the other ray is called the terminal side. The angle that is formed is identified by showing the direction and amount of rotation from the initial side to the terminal side. If the rotation is in the counterclockwise direction, the angle is positive; if the rotation is in the clockwise direction, the angle is negative. Lowercase Greek letters will be used to denote angles ($\alpha, \beta, \gamma, \theta$).

Counterclockwise:



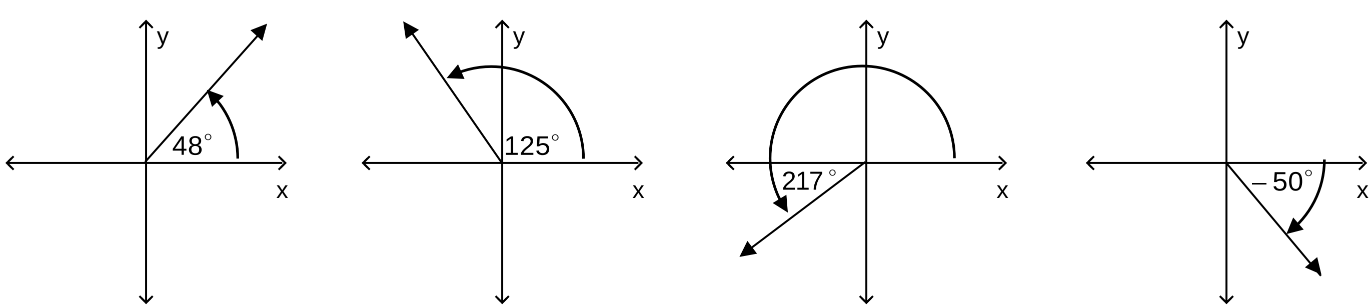
Clockwise:



An angle is in standard position in a Cartesian coordinate system if
1) its initial side is along the positive x-axis,
2) its vertex is at the origin, and
3) it is measured counterclockwise from the x-axis if its measure is positive, and clockwise if its measure is negative.

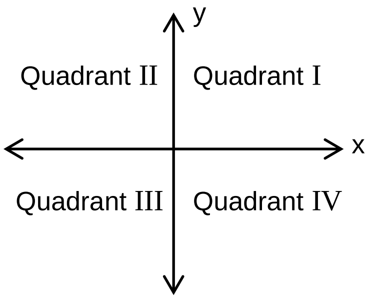
In the figures above, α and β are in standard position.

Example 1: Angles in standard position:



Section 1.3 – Measurement of Rotation – Day 1 (continued)

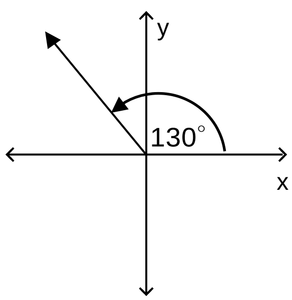
Recall that the Cartesian coordinate system is broken up into the following four quadrants:

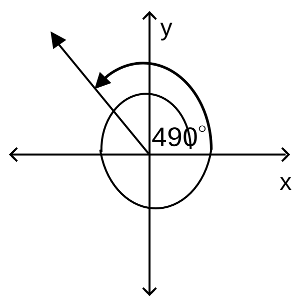


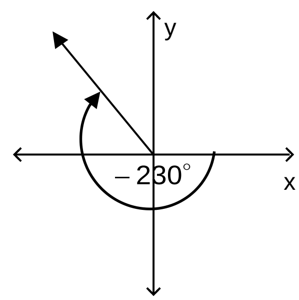
Use Roman numerals I, II, III, and IV to denote the four quadrants.

Two angles in standard position are coterminal if they terminate at the same place (i.e., the terminal rays coincide).

Example 2: Graph the angles 130°, 490°, and -230°. Determine whether they are coterminal angles.







Since the terminal side of each of the angles (130°, 490°, and -230°) is the same, the angles 130°, 490°, and -230° are coterminal angles. Notice that these coterminal angles differ by an integral multiple of 360° : $490^\circ = 130^\circ + 360^\circ$ and $-230^\circ = 130^\circ - 360^\circ$.

So, two angles θ and ϕ are coterminal if and only if $\phi = \theta + 360^\circ n$, where n is an integer.

Coterminal angles have the same initial side and the same terminal side, but different amounts of rotation.
To find a coterminal angle θ_c between 0° and 360° of an angle θ, add or subtract integral multiples of 360° to/from the original angle θ until you have an angle between 0° and 360°.

Example 3: Find a coterminal angle between 0° and 360° for each of the given angles.

a) 908° coterminal angle $\theta_c = 908^\circ - 2(360^\circ)$

$$= 908^\circ - 720^\circ$$
$$= 188^\circ$$

b) -830° The least integer multiple of 360° greater than 830° is $3(360^\circ) = 1080^\circ$. So, coterminal angle $\theta_c = 3(360^\circ) + (-830^\circ)$

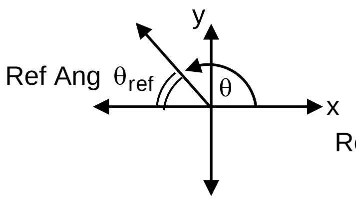
$$= 1080^\circ - 830^\circ$$
$$= 250^\circ$$

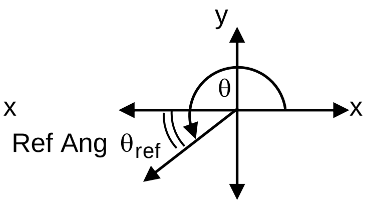
Section 1.3 – Measurement of Rotation – Day 1 (continued)

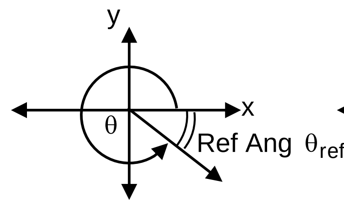
Reference Angles

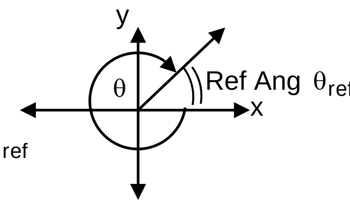
The reference angle of an angle in standard position is the positive, acute angle (or right angle) between the x-axis and the terminal side of the angle.

Let θ denote a nonacute angle that lies in a quadrant. The acute angle θ_{ref} ( ))) formed by the terminal side of θ and either the positive x-axis or the negative x-axis is called the reference angle for θ.



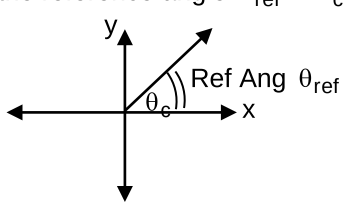




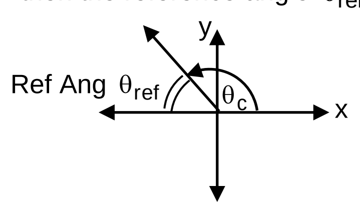


To find the reference angle θ_{ref} of a given angle θ, first find θ's coterminal angle between 0° and 360° and call it angle θ_c. Then the reference angle θ_{ref} can be calculated as follows, based on the location of the coterminal angle θ_c.

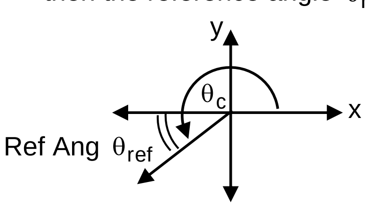
1) If the coterminal angle θ_c is in quadrant I, then the reference angle $\theta_{ref} = \theta_c$.



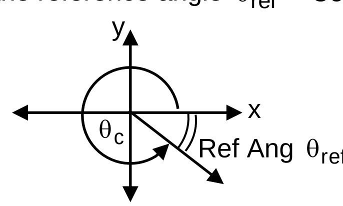
2) If the coterminal angle θ_c is in quadrant II, then the reference angle $\theta_{ref} = 180^\circ - \theta_c$.



3) If the coterminal angle θ_c is in quadrant III, then the reference angle $\theta_{ref} = \theta_c - 180^\circ$.



4) If the coterminal angle θ_c is in quadrant IV, then the reference angle $\theta_{ref} = 360^\circ - \theta_c$.



Example 4: Find the reference angle of each given angle.

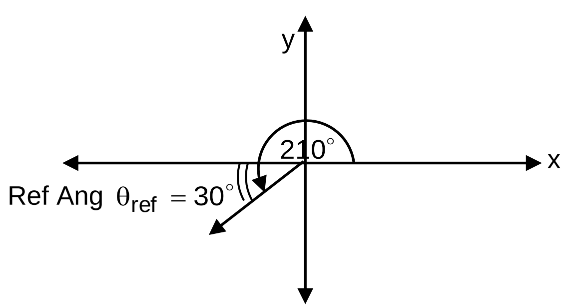
a) 210°

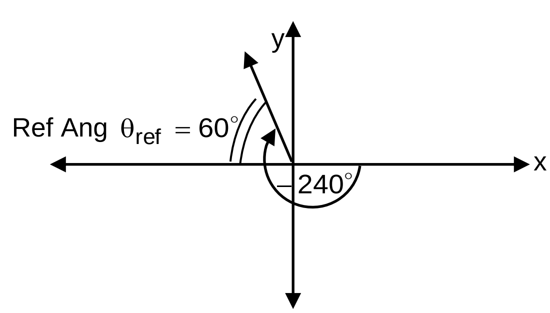
Coterminal angle $\theta_c = 210^\circ$

Quadrant III, so Ref Angle $\theta_{ref} = \theta_c - 180^\circ$
$= 210^\circ - 180^\circ$
$= 30^\circ$

b) -240° Coterminal angle $\theta_c = -240^\circ + 360^\circ$
$= 120^\circ$

Quadrant II, so Ref Angle $\theta_{ref} = 180^\circ - \theta_c$
$= 180^\circ - 120^\circ$
$= 60^\circ$





Section 1.3 – Measurement of Rotation – Day 1 (continued)

In summary, if θ_c is the degree measure of an angle between 0° and 360°, then the measure of its reference angle θ_{ref} can be calculated as follows:

If θ_c terminates in Quadrant I, $\theta_{ref} = \theta_c$.

If θ_c terminates in Quadrant II, $\theta_{ref} = 180^\circ - \theta_c$.

If θ_c terminates in Quadrant III, $\theta_{ref} = \theta_c - 180^\circ$.

If θ_c terminates in Quadrant IV, $\theta_{ref} = 360^\circ - \theta_c$.

If θ_c terminates on a quadrant boundary, $\theta_{ref} = 0^\circ$ or 90°.

The objective of drawing a picture of an angle in standard position reduces to the following procedure:

Task: To sketch an angle in standard position
1) If the angle is not between 0° and 360°, find a coterminal angle which is between 0° and 360°.
2) Find the measure of the reference angle.
3) Sketch the angle in the appropriate quadrant, using the reference angle to determine where in that quadrant it belongs.

Example 5: Sketch the angle $\theta = 5680^\circ$.

1) Find a coterminal angle between 0° and 360°.

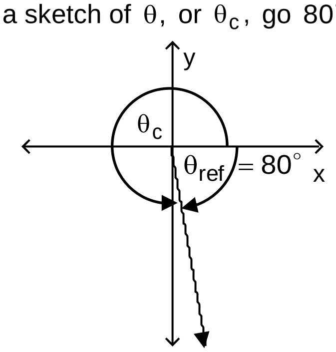
$\frac{5680}{360} \approx 15.777778 \Rightarrow$ 15 complete revolutions, then another 0.777778 of a revolution.

Thus,
$$\begin{aligned}\theta_c &= 5680^\circ - 15(360^\circ)\\ &= 5680^\circ - 5400^\circ\\ &= 280^\circ\end{aligned}$$

So, $\theta = 5680^\circ$ is coterminal with $\theta_c = 280^\circ$, which is in Quadrant IV.

2) Reference angle:
$$\begin{aligned}\theta_{ref} &= 360^\circ - \theta_c\\ &= 360^\circ - 280^\circ\\ &= 80^\circ\end{aligned}$$

3) To draw a sketch of θ, or θ_c, go 80° clockwise (below the x-axis) in the fourth quadrant.



All material has been taken from *Trigonometry,* by P. Foerster, 3rd Edition