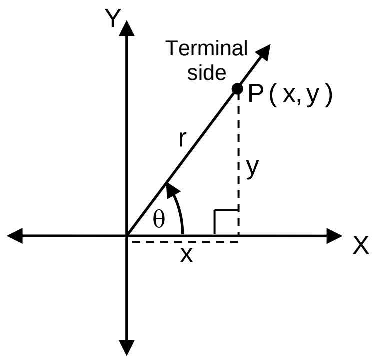
Section 1.4 – Definition of the Trigonometric Functions

Objectives: Be able to use the definitions to find exact values of the six trigonometric functions given
1) the coordinates of a point on the terminal side,
2) the value of one function and the quadrant in which the angle terminates,
3) a “special angle” whose measure is a multiple of 30° or 45°.

The position of the terminal side of an angle in standard position is uniquely determined by the measure of the angle. The six trigonometric functions provide numerical ways of describing the position of the terminal side of an angle in standard position.

Let θ be an angle in standard position in a Cartesian coordinate system. Let P, with coordinates (x, y), be a point on the terminal side. Thus, x and y can be considered to be the measures of the sides of the right triangle formed by drawing a perpendicular from P to the x-axis.



Definition: The six trigonometric functions – sine, cosine, tangent, cotangent, secant, and cosecant – are defined as follows:

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y} \qquad \sec\theta = \frac{r}{x} \qquad \csc\theta = \frac{r}{y}$$

In each of these trig function definitions, $\sin\theta$ for example, θ is the independent variable and sin is the function name. To be precise, we should write the function as $\sin(\theta)$, but if there is no confusion about the argument θ, the parentheses are often omitted.

We assume the r value is positive, since r represents the distance from the origin to the point (x, y).

Also remember our definitions of the six trigonometric functions using the right triangle ratios.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

We will get a lot of practice writing these six trigonometric ratios. Use the following technique:
1) Draw a picture of θ in standard position.
2) Pick a convenient point (x, y) on the terminal side.
3) Draw a perpendicular from (x, y) to the x-axis.
4) Fill in values for x, y, and r on the diagram.
5) Use the definitions to write values of the six trig functions.

Now that we have defined the six trig functions, you need to be able to find values of the functions for various values of angle θ.

Section 1.4 – Definition of the Trigonometric Functions (continued)

Objective 1 Examples:

Example 1: Find the six trigonometric functions of θ if the terminal side passes through the point $(4,-3)$.

$(4,-3)=(x,y) \quad \Rightarrow x=4, \quad y=-3$ and $(4,-3)$ is in Quadrant IV

$x^2+y^2=r^2$

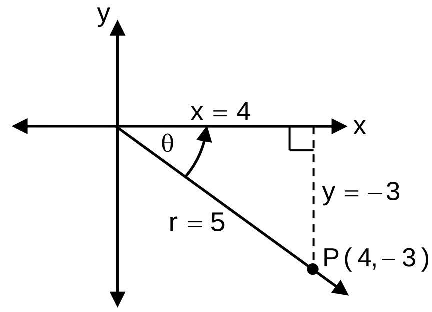
$r=\sqrt{x^2+y^2}$

$r=\sqrt{4^2+(-3)^2}$

$r=\sqrt{16+9}$

$r=\sqrt{25}$

$r=5$



$\sin\theta=\frac{y}{r}$ $\quad \cos\theta=\frac{x}{r}$ $\quad \tan\theta=\frac{y}{x}$ $\quad \cot\theta=\frac{x}{y}$ $\quad \sec\theta=\frac{r}{x}$ $\quad \csc\theta=\frac{r}{y}$

$\sin\theta=\frac{-3}{5}$ $\quad \cos\theta=\frac{4}{5}$ $\quad \tan\theta=\frac{-3}{4}$ $\quad \cot\theta=\frac{4}{-3}$ $\quad \sec\theta=\frac{5}{4}$ $\quad \csc\theta=\frac{5}{-3}$

Example 2: Find the six trigonometric functions of θ if the terminal side passes through the point $(-5,7)$.

$(-5,7)=(x,y) \quad \Rightarrow x=-5, \quad y=7$ and $(-5,7)$ is in Quadrant II

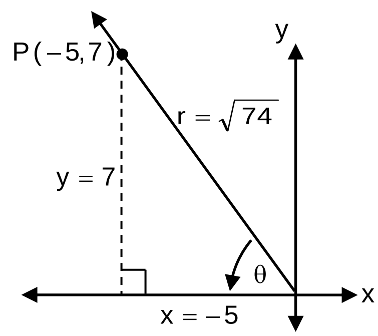
$x^2+y^2=r^2$

$r=\sqrt{x^2+y^2}$

$r=\sqrt{(-5)^2+(7)^2}$

$r=\sqrt{25+49}$

$r=\sqrt{74}$



$\sin\theta=\frac{y}{r}$ $\quad \cos\theta=\frac{x}{r}$ $\quad \tan\theta=\frac{y}{x}$ $\quad \cot\theta=\frac{x}{y}$ $\quad \sec\theta=\frac{r}{x}$ $\quad \csc\theta=\frac{r}{y}$

$\sin\theta=\frac{7}{\sqrt{74}}$ $\quad \cos\theta=\frac{-5}{\sqrt{74}}$ $\quad \tan\theta=\frac{7}{-5}$ $\quad \cot\theta=\frac{-5}{7}$ $\quad \sec\theta=\frac{\sqrt{74}}{-5}$ $\quad \csc\theta=\frac{\sqrt{74}}{7}$

$\sin\theta=\frac{7}{\sqrt{74}}\left(\frac{\sqrt{74}}{\sqrt{74}}\right)$ $\quad \cos\theta=\frac{-5}{\sqrt{74}}\left(\frac{\sqrt{74}}{\sqrt{74}}\right)$ $\quad \tan\theta=\frac{-7}{5}$ $\quad \sec\theta=\frac{-\sqrt{74}}{5}$

$\sin\theta=\frac{7\sqrt{74}}{74}$ $\quad \cos\theta=\frac{-5\sqrt{74}}{74}$

Section 1.4 – Definition of the Trigonometric Functions (continued)

Objective 2 Example:

Example 3: Find the six trigonometric functions of θ if θ terminates in Quadrant II and $\cos\theta = \frac{-2}{3}$.

$$\cos\theta = \frac{-2}{3}$$

$$= \frac{x}{r}$$

Quadrant $\text{II} \Rightarrow$ x is negative and y is positive; $\Rightarrow$ Assume $x = -2,\quad r = 3$

$$x^2 + y^2 = r^2$$

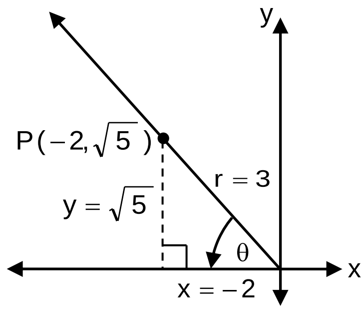
$$y = \sqrt{r^2 - x^2}$$

$$y = \sqrt{3^3 - (-2)^2}$$

$$y = \sqrt{9-4}$$

$$y = \sqrt{5}$$

So, $(x, y) = (-2, \sqrt{5})$



$\sin\theta = \frac{y}{r}$ $\quad\cos\theta = \frac{x}{r}$ $\quad\tan\theta = \frac{y}{x}$ $\quad\cot\theta = \frac{x}{y}$ $\quad\sec\theta = \frac{r}{x}$ $\quad\csc\theta = \frac{r}{y}$

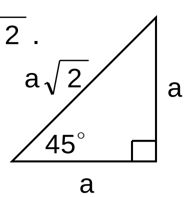
$\sin\theta = \frac{\sqrt{5}}{3}$ $\quad\cos\theta = \frac{-2}{3}$ $\quad\tan\theta = \frac{\sqrt{5}}{-2}$ $\quad\cot\theta = \frac{-2}{\sqrt{5}}$ $\quad\sec\theta = \frac{3}{-2}$ $\quad\csc\theta = \frac{3}{\sqrt{5}}$

$\tan\theta = \frac{-\sqrt{5}}{2}$ $\quad\cot\theta = \frac{-2}{\sqrt{5}}\left(\frac{\sqrt{5}}{\sqrt{5}}\right)$ $\quad\sec\theta = \frac{-3}{2}$ $\quad\csc\theta = \frac{3}{\sqrt{5}}\left(\frac{\sqrt{5}}{\sqrt{5}}\right)$

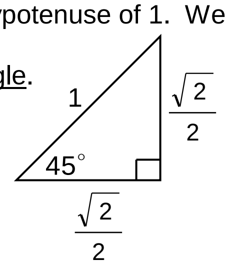
$\cot\theta = \frac{-2\sqrt{5}}{5}$ $\quad\csc\theta = \frac{3\sqrt{5}}{5}$

From Chapter 0, recall the $45^\circ - 45^\circ - 90^\circ$ triangle and $30^\circ - 60^\circ - 90^\circ$ triangle relationships.

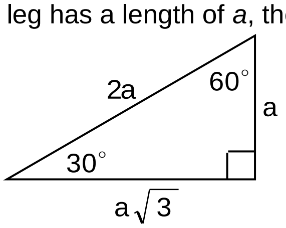
A $45^\circ - 45^\circ - 90^\circ$ triangle is an isosceles right triangle with congruent legs. If the length of a leg is *a*, then the length of the hypotenuse is $a\sqrt{2}$.



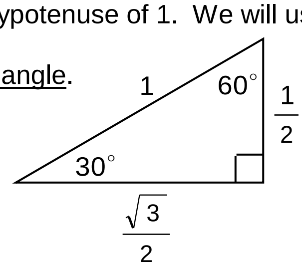
If we let $a = \frac{\sqrt{2}}{2}$, then we get a hypotenuse of 1. We will use this as our standard $45^\circ - 45^\circ - 90^\circ$ triangle.



In a $30^\circ - 60^\circ - 90^\circ$ triangle, the shorter leg is opposite the 30° angle and the longer leg is opposite the 60° angle. If the shorter leg has a length of *a*, then the hypotenuse has length 2a and the longer leg has length $a\sqrt{3}$.



If we let $a = \frac{1}{2}$, then we get a hypotenuse of 1. We will use this as our standard $30^\circ - 60^\circ - 90^\circ$ triangle.



Section 1.4 – Definition of the Trigonometric Functions (continued)

Objective 3 Example:

Example 4: Find the six trigonometric functions of $\theta = 210^\circ$.

$\theta = 210^\circ \;\Rightarrow \theta$ terminates in Quadrant III

$\theta_c = 210^\circ$

Reference angle, $\theta_{ref} = \theta_c - 180^\circ$

$= 210^\circ - 180^\circ$

$= 30^\circ$

Using our $30^\circ - 60^\circ - 90^\circ$ special triangle relationship, let the point for 210° be $(x, y) = \left(\frac{-\sqrt{3}}{2}, \frac{-1}{2}\right)$.

Find r: $x^2 + y^2 = r^2$

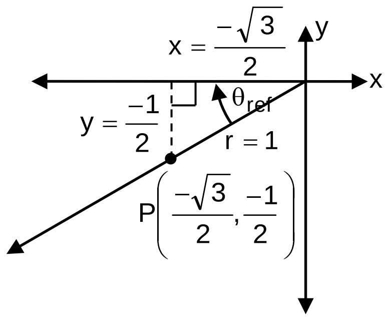
$$r = \sqrt{x^2 + y^2}$$

$$r = \sqrt{\left(\frac{-\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$

$$r = \sqrt{\frac{3}{4} + \frac{1}{4}}$$

$$r = \sqrt{1}$$

$$r = 1$$



$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y} \qquad \sec\theta = \frac{r}{x} \qquad \csc\theta = \frac{r}{y}$$

$$\sin\theta = \frac{\frac{-1}{2}}{1} \qquad \cos\theta = \frac{\frac{-\sqrt{3}}{2}}{1} \qquad \tan\theta = \frac{\frac{-1}{2}}{\frac{-\sqrt{3}}{2}} \qquad \cot\theta = \frac{\frac{-\sqrt{3}}{2}}{\frac{-1}{2}} \qquad \sec\theta = \frac{1}{\frac{-\sqrt{3}}{2}} \qquad \csc\theta = \frac{1}{\frac{-1}{2}}$$

$$\sin\theta = \frac{-1}{2} \qquad \cos\theta = \frac{-\sqrt{3}}{2} \qquad \tan\theta = \frac{1}{\sqrt{3}} \qquad \cot\theta = \frac{\sqrt{3}}{1} \qquad \sec\theta = \frac{-2}{\sqrt{3}} \qquad \csc\theta = -2$$

$$\tan\theta = \frac{1}{\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3}}\right) \qquad \cot\theta = \sqrt{3} \qquad \sec\theta = \frac{-2}{\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3}}\right)$$

$$\tan\theta = \frac{\sqrt{3}}{3} \qquad \sec\theta = \frac{-2\sqrt{3}}{3}$$