Section 2.7 – Graphs of Circular Function Sinusoids

Objective: Given the equation of a sinusoid involving circular functions, sketch the graph, and vice versa.

In Section 2.4, you learned that the general equation for a sinusoidal function is of the form $y = VD + A\sin\left[B(\theta - PD)\right]$ or $y = VD + A\cos\left[B(\theta - PD)\right]$. The independent variable is θ and the dependent variable is y. The constants A, B, VD, and PD determine the shape and placement of the graph. The general equation of a circular sinusoidal function has exactly the same form except that x is used for the independent variable rather than θ. So the equations for circular sinusoidal functions are $y = VD + A\sin\left[B(x - PD)\right]$ or $y = VD + A\cos\left[B(x - PD)\right]$. The x stands for a number rather than an angle, so the functions are more suitable to the real-world than trigonometric functions. Instead of an angle θ, the independent variable x may, for example, represent time or distance.

The only difference between the graphs of circular functions and trigonometric functions is the fact that circular sine and cosine functions make a complete cycle for each $2\pi-\text{unit}$ change in the argument rather than the $360^\circ-\text{unit}$ change in the argument for trigonometric functions. So for circular functions, the period is given by $\text{Period} = \frac{2\pi}{B}$, instead of $\text{Period} = \frac{360^\circ}{B}$.

In a circular function, if the coefficient B is a multiple of π, then the period will be a rational number. If B is not a multiple of π, then the period will be a multiple of π. If B is not a multiple of π, then a good way to draw the graph is to mark the x-axis with a scale in multiples of π.

Section 2.7 – Graphs of Circular Function Sinusoids (continued)

Example 1: Find the period, amplitude, frequency, phase displacement, and vertical displacement. Then use this information to find critical points and sketch the graph.

$$y = 2 + 5\cos\left[\frac{\pi}{4}(x-3)\right]$$

$$y = VD + A\cos\left[B(x - PD)\right]$$

$$\Rightarrow\ A = 5,\ B = \frac{\pi}{4},\ VD = 2,\ \text{and}\ PD = 3$$

$$\text{Amplitude} = |A| = |5| = 5$$

$$\text{Period } P = \frac{2\pi}{B} = \frac{2\pi}{\frac{\pi}{4}} = 2\pi\left(\frac{4}{\pi}\right) = 8$$

$$\text{Frequency} = \frac{1}{\text{Period}} = \frac{1}{8}\ \text{cycle per unit}$$

$$x_{step} = \frac{\text{Period}}{4} = \frac{8}{4} = 2$$

Phase displacement: $PD = 3\ \Rightarrow$ shift right 3 units

Vertical displacement: $VD = 2\ \Rightarrow$ shift up 2 units, so the sinusoidal axis is at $y = 2$

You want to find values of x that make the argument equal to $0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2}$, and 2π. So, set

$$\frac{\pi}{4}(x-3) = 0 \qquad x - 3 = 0 \qquad x = 3$$

$$\frac{\pi}{4}(x-3) = \frac{\pi}{2} \qquad x - 3 = \frac{\pi}{2}\left(\frac{4}{\pi}\right) \qquad x - 3 = \frac{4}{2} \qquad x - 3 = 2 \qquad x = 5$$

$$\frac{\pi}{4}(x-3) = \pi \qquad x - 3 = \pi\left(\frac{4}{\pi}\right) \qquad x - 3 = 4 \qquad x = 7$$

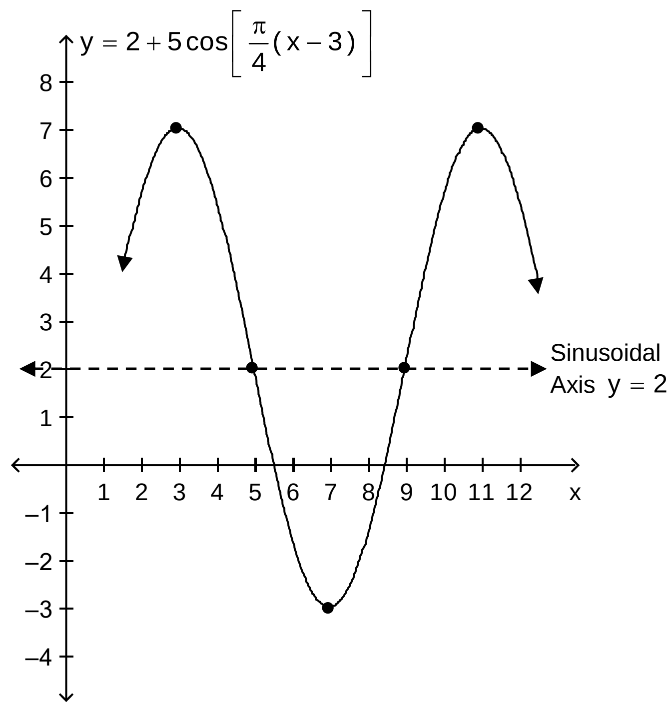
$$\frac{\pi}{4}(x-3) = \frac{3\pi}{2} \qquad x - 3 = \frac{3\pi}{2}\left(\frac{4}{\pi}\right) \qquad x - 3 = \frac{12}{2} \qquad x - 3 = 6 \qquad x = 9$$

$$\frac{\pi}{4}(x-3) = 2\pi \qquad x - 3 = 2\pi\left(\frac{4}{\pi}\right) \qquad x - 3 = 8 \qquad x = 11$$

Critical Points

x	$\frac{\pi}{4}(x-3)$	$y = 2 + 5\cos\left[\frac{\pi}{4}(x-3)\right]$
3	0	7
5	$\frac{\pi}{2}$	2
7	π	-3
9	$\frac{3\pi}{2}$	2
11	2π	7

Remember, $x_{step} = 2$, so the x-values are 2 units apart.



Section 2.7 – Graphs of Circular Function Sinusoids (continued)

Example 2: Find the period, amplitude, frequency, phase displacement, and vertical displacement. Then use this information to find critical points and sketch the graph.

$$y = -3 + 4\sin\left[\frac{1}{2}(x+\pi)\right]$$

$$y = VD + A\sin\left[B(x - PD)\right]$$

$$\Rightarrow \quad A = 4,\ B = \frac{1}{2},\ VD = -3,\ \text{and } PD = -\pi$$

Amplitude $= |A| = |4| = 4$

Period $P = \dfrac{2\pi}{B} = \dfrac{2\pi}{\frac{1}{2}} = 2\pi\left(\dfrac{2}{1}\right) = 4\pi$

Frequency $= \dfrac{1}{\text{Period}} = \dfrac{1}{4\pi}$ cycle per unit

$x_{step} = \dfrac{\text{Period}}{4} = \dfrac{4\pi}{4} = \pi$

Phase displacement: $PD = -\pi \Rightarrow$ shift left π units

Vertical displacement: $VD = -3 \Rightarrow$ shift down 3 units, so the sinusoidal axis is at $y = -3$

You want to find values of x that make the argument equal to $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π. So, set

$\frac{1}{2}(x+\pi) = 0$; $x + \pi = 0$; $x = -\pi$

$\frac{1}{2}(x+\pi) = \frac{\pi}{2}$; $x + \pi = \frac{\pi}{2}\left(\frac{2}{1}\right)$; $x + \pi = \pi$; $x = 0$

$\frac{1}{2}(x+\pi) = \pi$; $x + \pi = \pi\left(\frac{2}{1}\right)$; $x + \pi = 2\pi$; $x = \pi$

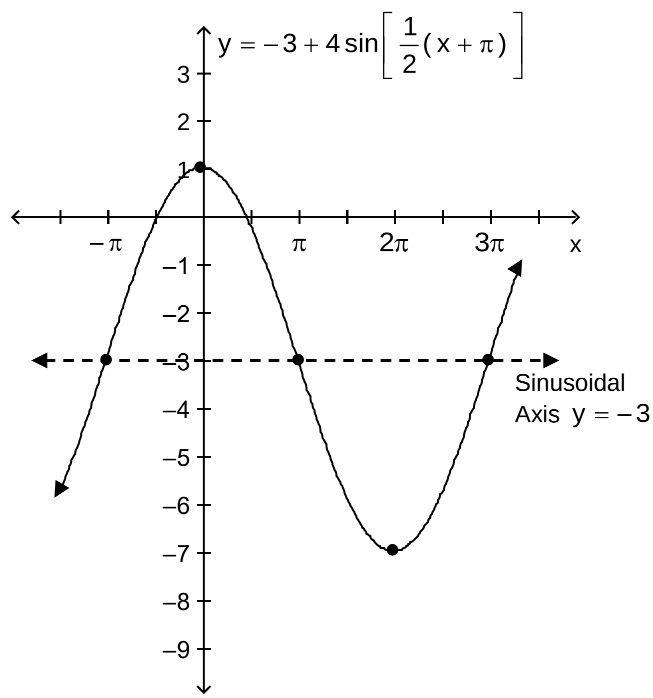
$\frac{1}{2}(x+\pi) = \frac{3\pi}{2}$; $x + \pi = \frac{3\pi}{2}\left(\frac{2}{1}\right)$; $x + \pi = 3\pi$; $x = 2\pi$

$\frac{1}{2}(x+\pi) = 2\pi$; $x + \pi = 2\pi\left(\frac{2}{1}\right)$; $x + \pi = 4\pi$; $x = 3\pi$

Critical Points

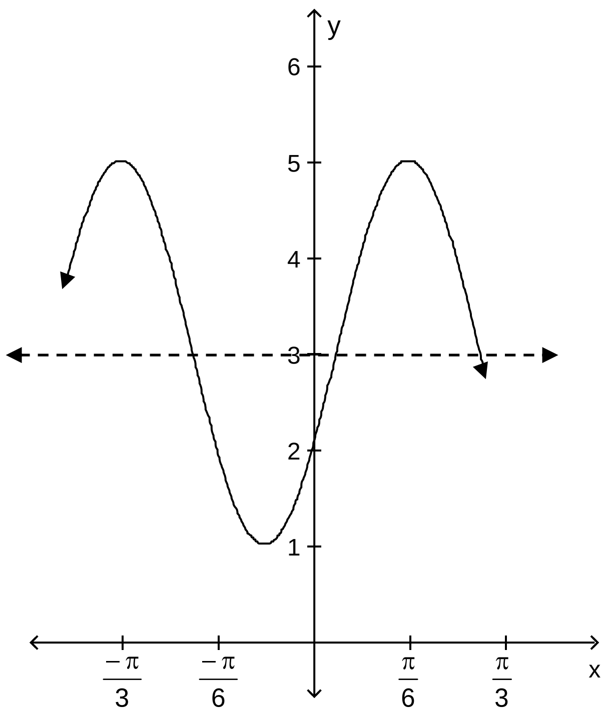
x	$\frac{1}{2}(x+\pi)$	$y = -3 + 4\sin\left[\frac{1}{2}(x+\pi)\right]$
$-\pi$	0	-3
0	$\frac{\pi}{2}$	1
π	π	-3
2π	$\frac{3\pi}{2}$	-7
3π	2π	-3

Remember, $x_{step} = \pi$, so the x-values are π units apart.



Section 2.7 – Graphs of Circular Function Sinusoids (continued)

Example 3: Write an equation of the sinusoid sketched, using a circular function. Determine the period, amplitude, frequency, phase displacement, and vertical displacement.



Assume it is a non-reflected cosine curve with

Phase Displacement $PD = \frac{-\pi}{3}$

Upper Bound $= 5$

Sinusoidal Axis: $y = 3$, so $\boxed{VD = 3}$

$$\text{Amplitude} = \text{Upper Bound} - \text{Sinusoidal Axis}$$
$$= 5 - 3$$
$$= 2$$

A non-reflected cosine curve $\Rightarrow \boxed{A = 2}$

$$\text{Period} = \frac{\pi}{6} - \left(\frac{-\pi}{3}\right)$$
$$= \frac{\pi}{6} + \frac{\pi}{3}$$
$$= \frac{\pi}{6} + \frac{2\pi}{6}$$
$$= \frac{3\pi}{6}$$
$$= \frac{\pi}{2}$$

$$\text{So, Period} = \frac{2\pi}{B}$$
$$\Rightarrow \frac{\pi}{2} = \frac{2\pi}{B}$$
$$B\pi = 2(2\pi)$$
$$B\pi = 4\pi$$
$$\boxed{B = 4}$$

$$\text{Frequency} = \frac{1}{\text{Period}}$$
$$= \frac{1}{\frac{\pi}{2}}$$
$$= \frac{2}{\pi} \text{ cycle per unit}$$

$$\text{Equation: } y = VD + A\cos\left[B(x - PD)\right]$$
$$y = 3 + 2\cos\left[4\left(x - \frac{-\pi}{3}\right)\right]$$
$$y = 3 + 2\cos\left[4\left(x + \frac{\pi}{3}\right)\right]$$

All material has been taken from *Trigonometry,* by P. Foerster, 3rd Edition