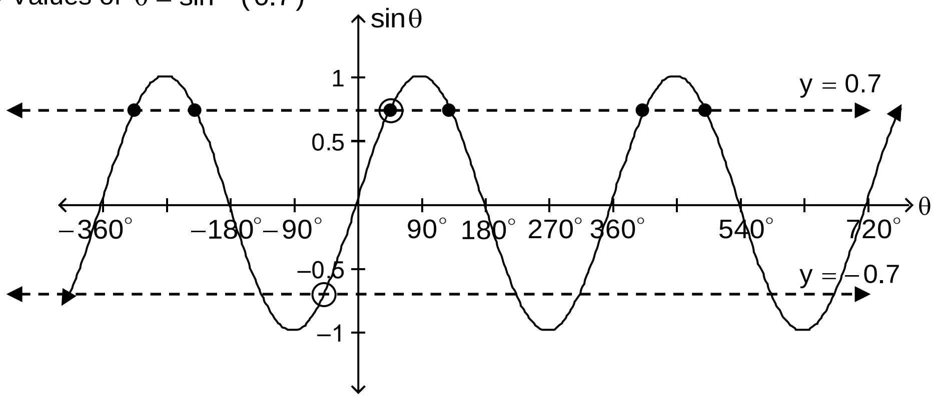
Section 2.10 – General Inverse Sine and Cosine Relations

The definition of $\theta = \sin^{-1}(0.7)$ tells you that θ is "an angle whose sine is 0.7." As you can see from the graph of the sine function below, there are many angles whose sine is 0.7.

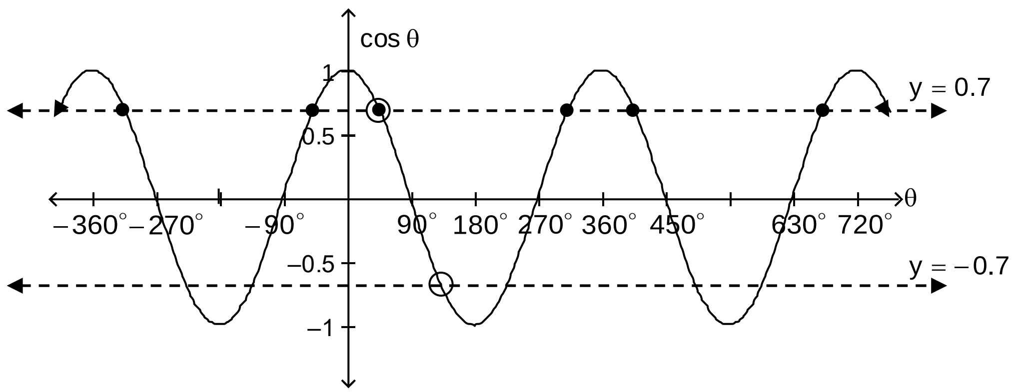
- ⇒ Values of $\theta = \sin^{-1}(0.7)$



$\sin\theta$ is positive in Quadrants I and II and negative in Quadrants III and IV.

Similarly, there are an infinite number of values of $\theta = \cos^{-1}(0.7)$, as shown below.

- ⇒ Values of $\theta = \cos^{-1}(0.7)$



$\cos\theta$ is positive in Quadrants I and IV and negative in Quadrants II and III.

In this section, we will learn how to find any desired value of an inverse trigonometric or circular relation.

Objective: Be able to find values of θ or x for $\sin^{-1}(y)$ and $\cos^{-1}(y)$.

Find $\theta = \sin^{-1}(0.7)$ and $\theta = \cos^{-1}(0.7)$ on your calculator:

$\theta = \sin^{-1}(0.7) \approx 44.43^\circ$ $\qquad$ $\theta = \cos^{-1}(0.7) \approx 45.57^\circ$

If the argument is negative, something surprising happens:

$\theta = \sin^{-1}(-0.7) \approx -44.43^\circ$ $\qquad$ $\theta = \cos^{-1}(-0.7) \approx 134.43^\circ$

$\theta = \sin^{-1}(-0.7)$ is a fourth quadrant angle, but $\theta = \cos^{-1}(-0.7)$ is a second quadrant angle. You can see why from the graphs above (note ○s on graphs above).

On the calculator, $\sin^{-1}(y)$ is always an angle from -90° to 90° (or $\frac{-\pi}{2}$ to $\frac{\pi}{2}$). $\cos^{-1}(y)$ is always an angle from 0° to 180° (or 0 to π). Values of $\sin^{-1}(y)$ and $\cos^{-1}(y)$ in these ranges are called the <u>principal values</u> of the inverse sine and inverse cosine relations. To distinguish between the principal values and any other values of $\sin^{-1}(y)$ and $\cos^{-1}(y)$, a <u>capital letter</u> is used for the name, $\text{Sin}^{-1}(y)$ and $\text{Cos}^{-1}(y)$.

Section 2.10 – General Inverse Sine and Cosine Relations (continued)

Definition: Principal values of inverse trigonometric relations:

$\theta = \text{Sin}^{-1}(y)$ means $y = \sin\theta$ and $-90^\circ \le \theta \le 90^\circ$

$\theta = \text{Cos}^{-1}(y)$ means $y = \cos\theta$ and $0^\circ \le \theta \le 180^\circ$

Principal values of inverse circular relations:

$x = \text{Sin}^{-1}(y)$ means $y = \sin x$ and $\frac{-\pi}{2} \le x \le \frac{\pi}{2}$

$x = \text{Cos}^{-1}(y)$ means $y = \cos x$ and $0 \le x \le \pi$

Example 1: Find all values of $\theta = \cos^{-1}(-0.2)$.

From your calculator, you get $\theta = \text{Cos}^{-1}(-0.2)$ (Degree Mode)

$= 101.536^\circ$

$\approx 101.54^\circ$ Quadrant II

From the cosine graph above, you can see that $\cos(-101.54^\circ)$ also equals -0.2.

So, $\theta = \cos^{-1}(-0.2)$

$\approx \pm 101.54^\circ$

All of the other angles will be coterminal with these two values. A compact way of writing the general solution is $\theta = \cos^{-1}(-0.2)$

$\approx \pm 101.54^\circ + 360^\circ n$, where n is an integer.

Example 2: For $\theta = \cos^{-1}(0.68)$, find θ to 2 decimal places, getting:

a) the general solution
b) the first three positive values of θ

From your calculator, you get $\theta = \text{Cos}^{-1}(0.68)$

$= 47.156^\circ$

$\approx 47.16^\circ$ Quadrant I

a) The general solution (GS): $\theta \approx \pm 47.16^\circ + 360^\circ n$, where n is an integer.

b) $\theta_+ : 47.16^\circ,\ 407.16^\circ,\ 767.16^\circ, \ldots$

$\theta_- : 312.84^\circ,\ 672.84^\circ,\ 1032.84^\circ, \ldots$

So, the first three positive values of θ are $\theta = 47.16^\circ, 312.84^\circ, 407.16^\circ$.

Example 3: Find all values of $x = \sin^{-1}(0.72)$.

From your calculator, you get $x = \text{Sin}^{-1}(0.72)$ (Radian Mode)

$= 0.80380$

≈ 0.8038 Quadrant I

From the sine graph above, you know sine is also positive in Quadrant II.

So, $x = \sin^{-1}(0.72)$

≈ 0.8038 and $\pi - 0.8038 \approx 2.33779$

≈ 2.3378

All of the other angles will be coterminal with these two values. A compact way of writing the general solution is $x = \sin^{-1}(0.72)$

$x \approx 0.8038 + 2\pi n$ or $x \approx 2.3378 + 2\pi n$, where n is an integer.

Section 2.10 – General Inverse Sine and Cosine Relations (continued)

Example 4: For $x = \sin^{-1}(0.83)$, find x to 4 decimal places, getting:

a) the general solution
b) the first three positive values of x

From your calculator, you get $x = \text{Sin}^{-1}(0.83)$ (Radian Mode)

$= 0.97910$

≈ 0.9791 Quadrant I

a) The general solution (GS): $x_1 \approx 0.9791 + 2\pi n$ or $x_2 \approx (\pi - 0.9791) + 2\pi n$, where n is an integer.

$x_1 \approx 0.9791 + 2\pi n$ or $x_2 \approx 2.1625 + 2\pi n$, where n is an integer.

b) 1^{st} : 0.9791, 7.2623 , 13.5455 , …

2^{nd} : 2.1625, 8.4457 , 14.7289 , …

So, the first three positive values of x are $x = 0.9791, 2.1625, 7.2623$.

Example 5: Find all values of $x = \cos^{-1}(0.49)$.

From your calculator, you get $x = \text{Cos}^{-1}(0.49)$ (Radian Mode)

$= 1.05870$

≈ 1.0587 Quadrant I

Cosine is also positive in Quadrant IV, so $\cos(-1.0587)$ also equals 0.49 .

So, $x = \cos^{-1}(0.49)$

$\approx \pm 1.0587$

All of the other angles will be coterminal with these two values. A compact way of writing the general solution is $x = \cos^{-1}(0.49)$

$\approx \pm 1.0587 + 2\pi n$, where n is an integer.

Example 6: For $\theta = \sin^{-1}(-0.42)$, find θ to 2 decimal places, getting:

a) the general solution
b) the first three positive values of θ

From your calculator, you get $\theta = \text{Sin}^{-1}(-0.42)$ (Degree Mode)

$= -24.834^\circ$

$\approx -24.83^\circ$ Quadrant IV

a) The general solution (GS):

Sine is also negative in Quadrant III, so

$\theta_1 \approx -24.83^\circ + 360^\circ n$ or $\theta_2 \approx 180^\circ - (-24.83^\circ) + 360^\circ n$, where n is an integer.

$\theta_1 \approx -24.83^\circ + 360^\circ n$ or $\theta_2 \approx 204.83^\circ + 360^\circ n$, where n is an integer.

b) 1^{st} : -24.83° , 335.17° , 695.17° , …

2^{nd} : 204.83° , 564.83° , 924.83° , …

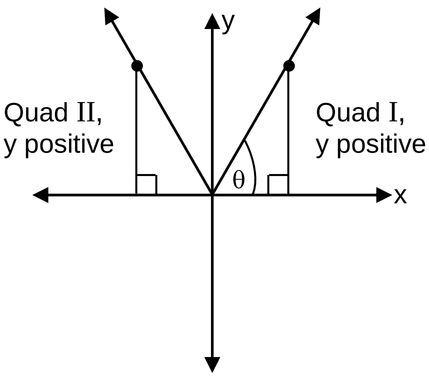
So, the first three positive values of θ are $\theta = 204.83^\circ, 335.17^\circ, 564.83^\circ$.

Section 2.10 – General Inverse Sine and Cosine Relations (continued)

Another way to look at these problems:

Remember that $\sin\theta = \dfrac{y}{r}$ and $\cos\theta = \dfrac{x}{r}$.

Where are the y-values the same or the x-values the same?

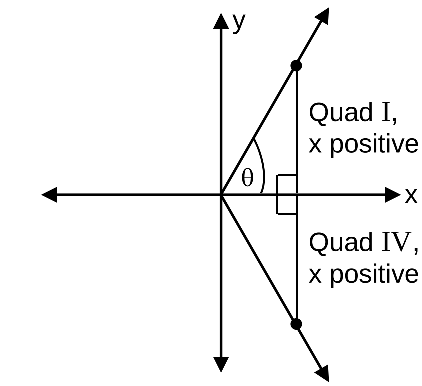


$\sin^{-1}(A)$:

$\theta + 360^\circ n$ or

$(180^\circ - \theta) + 360^\circ n$

Or radian mode,
$x + 2\pi n$ or
$(\pi - x) + 2\pi n$

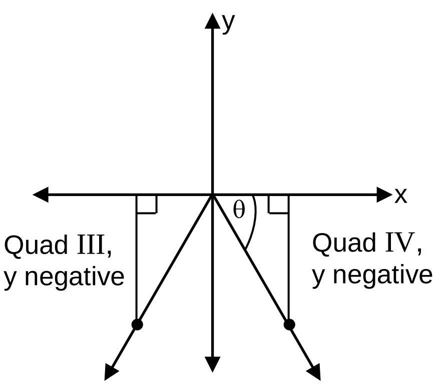


$\cos^{-1}(A)$:

$\theta + 360^\circ n$ or

$-\theta + 360^\circ n$

Or radian mode,
$x + 2\pi n$ or
$-x + 2\pi n$

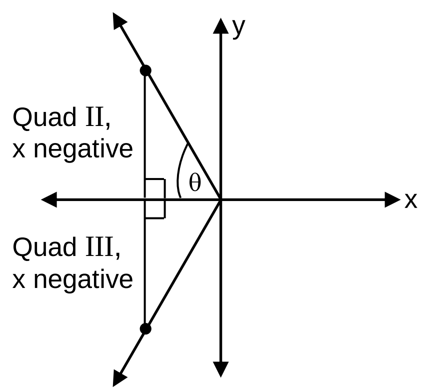


$\sin^{-1}(A)$:

$\theta + 360^\circ n$ or

$(180^\circ - \theta) + 360^\circ n$

Or radian mode,
$x + 2\pi n$ or
$(\pi - x) + 2\pi n$



$\cos^{-1}(A)$:

$\theta + 360^\circ n$ or

$-\theta + 360^\circ n$

Or radian mode,
$x + 2\pi n$ or
$-x + 2\pi n$

Summary:

$\theta = \text{Sin}^{-1}(\text{decimal})$ means principal value: $-90^\circ \le \theta \le 90^\circ$

$x = \text{Sin}^{-1}(\text{decimal})$ means principal value: $\dfrac{-\pi}{2} \le x \le \dfrac{\pi}{2}$

$\theta = \text{Cos}^{-1}(\text{decimal})$ means principal value: $0^\circ \le \theta \le 180^\circ$

$x = \text{Cos}^{-1}(\text{decimal})$ means principal value: $0 \le x \le \pi$

$\theta = \sin^{-1}(\text{decimal})$ means all values: $(\text{principal value} + 360^\circ n)$ and $(180^\circ - \text{principal value} + 360^\circ n)$, for n any integer

$x = \sin^{-1}(\text{decimal})$ means all values: $(\text{principal value} + 2\pi n)$ and $(\pi - \text{principal value} + 2\pi n)$, for n any integer

$\theta = \cos^{-1}(\text{decimal})$ means all values: $\pm\text{principal value} + 360^\circ n$, for n any integer

$x = \cos^{-1}(\text{decimal})$ means all values: $\pm\text{principal value} + 2\pi n$, for n any integer

All material has been taken from *Trigonometry,* by P. Foerster, 3rd Edition