Section 3.1 – Three Properties of Trigonometric Functions

We will now turn our focus from graphs of trigonometric functions to simplifying expressions with trigonometric functions.

In this chapter you will learn how to solve more complicated trigonometric equations with one variable. Some of these equations will involve the trigonometric or circular functions. Others will involve functions with more than one argument. Solving these equations requires you to learn some properties relating the trigonometric functions to one another. Using these properties, you will be able to transform, for example, sines to cosines, functions of 3x to functions of x, and functions of $(x+45^\circ)$ to functions of x and of 45°. These simplifying transformations are what you need to solve this type of equation. Solutions of these equations are inverse trigonometric or circular functions, which you studied in Chapters 1 and 2.

Section 3.1 – Three Properties of Trigonometric Functions

Objective: Use the reciprocal, quotient, and Pythagorean properties to transform given expressions to equivalent, simpler forms.

There are three types of properties of the trigonometric functions, which come directly from their definitions.

1) Reciprocal Properties

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

So, if you see $\csc x$ in a problem, you can replace it with $\frac{1}{\sin x}$.

Since the product of a number and its reciprocal equals 1, if you solve the equations above to get 1 by itself, you get the following modified reciprocal properties:

$$\sin x \csc x = 1 \qquad \cos x \sec x = 1 \qquad \tan x \cot x = 1$$

2) Quotient Properties

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

Example 1: Transform the expression on the left to the one on the right.
$\sin x \cot x$ to $\cos x$

$$\sin x \cot x = \sin x \frac{\cos x}{\sin x} \quad \text{Quotient property}$$
$$= \frac{\cancel{\sin x}\cos x}{\cancel{\sin x}} \quad \text{Multiplication property of fractions}$$
$$= \cos x$$
$$\therefore \sin x \cot x = \cos x$$

Example 2: Transform the expression on the left to the one on the right.
$\sin x \sec x \cot x$ to 1

$$\sin x \sec x \cot x = \sin x\left(\frac{1}{\cos x}\right)\left(\frac{\cos x}{\sin x}\right) \quad \text{Reciprocal and quotient properties}$$
$$= \frac{\cancel{\sin x}\,\cancel{\cos x}}{\cancel{\cos x}\,\cancel{\sin x}} \quad \text{Multiplication property of fractions}$$
$$= 1$$
$$\therefore \sin x \sec x \cot x = 1$$

Section 3.1 – Three Properties of Trigonometric Functions (continued)

Example 3: Transform the expression on the left to the one on the right.

a) $\csc\theta\cos\theta$ to $\cot\theta$

$\csc\theta\cos\theta =$

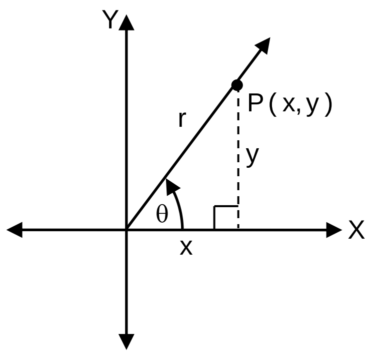
b) $\cot x$ to $\dfrac{\csc x}{\sec x}$

$\cot x =$

3) Pythagorean Properties

$$\sin^2 x + \cos^2 x = 1 \qquad 1 + \tan^2 x = \sec^2 x \qquad \cot^2 x + 1 = \csc^2 x$$

These properties are called the Pythagorean Properties, because they come from the Pythagorean Theorem.



$$x^2 + y^2 = r^2$$
$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$
$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$
$$\cos^2\theta + \sin^2\theta = 1$$
$$\sin^2\theta + \cos^2\theta = 1$$

If $\cos\theta \neq 0$, then dividing $\sin^2\theta + \cos^2\theta = 1$ by $\cos^2\theta$ gives

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$
$$\left(\frac{\sin\theta}{\cos\theta}\right)^2 + 1 = \left(\frac{1}{\cos\theta}\right)^2$$
$$\tan^2\theta + 1 = \sec^2\theta$$

Similarly, if $\sin\theta \neq 0$ and you divide $\sin^2\theta + \cos^2\theta = 1$ by $\sin^2\theta$, you get:

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$
$$1 + \left(\frac{\cos\theta}{\sin\theta}\right)^2 = \left(\frac{1}{\sin\theta}\right)^2$$
$$1 + \cot^2\theta = \csc^2\theta$$

Section 3.1 – Three Properties of Trigonometric Functions (continued)

Example 4: Transform the expression on the left to the one on the right.

$\cos^2 A - \sin^2 A$ to $1 - 2\sin^2 A$

$\cos^2 A - \sin^2 A =$

Example 5: Transform the expression on the left to the one on the right.

a) $\tan\theta(\sin\theta + \cot\theta\cos\theta)$ to $\sec\theta$

$\tan\theta(\sin\theta + \cot\theta\cos\theta) =$

b) $\csc x - \sin x$ to $\cot x \cos x$

$\csc x - \sin x =$

Remember that there is often more than one way to solve these problems.

All material has been taken from *Trigonometry,* by P. Foerster, 3rd Edition