Section 3.2 – Trigonometric Identities and Equations

An "open sentence" is another term for an equation. For an open sentence such as $\tan\theta = 1$, only certain values of θ make it true ($\theta = 45^\circ$, $\theta = 225^\circ$, or any other angles coterminal with these angles). So, $\tan\theta = 1$ is called a conditional equation, since it is only true under certain conditions. Some open sentences are true under all conditions, such as $\sin^2\theta + \cos^2\theta = 1$. $\sin^2\theta + \cos^2\theta = 1$ is true for all values of θ. This equation is called an identity because the two sides are equivalent expressions, i.e. they are "identical" to each other.

In this section, you will be given an equation, and you have to prove that the equation is an identity by showing that the two sides are identical. It is best to start with the more complicated (messy) side.

Objective: Given a trigonometric equation, prove that it is an identity.

There are two purposes for learning how to prove identities.
1) To learn the relationship among the functions.
2) To learn to transform one trigonometric expression to another equivalent form, usually simplifying it.

To accomplish this objective, the following agreement is made.
Agreement: To prove that an equation is an identity, start with one side and transform it into the other side.

So proving identities and transforming expressions (Sec 3.1) are equivalent problems. The only thing that is new is that you are allowed to pick the side you want to start with and then show it equals the other side.

Example 1: Prove that $(1+\cos x)(1-\cos x) = \sin^2 x$.

$$\begin{aligned}(1+\cos x)(1-\cos x) &= 1 - \cos x + \cos x - \cos^2 x && \text{FOIL}\\ &= 1-\cos^2 x && \text{Simplify}\\ &= \sin^2 x && \text{Familiar expression? Pythagorean property}\end{aligned}$$
$$\therefore (1+\cos x)(1-\cos x) = \sin^2 x$$

Example 2: Prove that $\cot x + \tan x = \csc x \sec x$.

$$\begin{aligned}\cot x + \tan x &= \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} && \text{Quotient properties}\\ &= \frac{\cos x}{\sin x}\left(\frac{\cos x}{\cos x}\right) + \frac{\sin x}{\cos x}\left(\frac{\sin x}{\sin x}\right) && \text{Common denominator}\\ &= \frac{\cos^2 x + \sin^2 x}{\sin x \cos x} && \text{Combine the fractions into one fraction}\\ &= \frac{1}{\sin x \cos x} && \text{Pythagorean property}\\ &= \frac{1}{\sin x}\left(\frac{1}{\cos x}\right) && \text{Multiplication property}\\ &= \csc x \sec x && \text{Reciprocal properties}\end{aligned}$$
$$\therefore \cot x + \tan x = \csc x \sec x$$

Section 3.2 – Trigonometric Identities and Equations (continued)

Example 3: Prove that $\csc\theta\cos^2\theta+\sin\theta=\csc\theta$ two different ways.

1) $\csc\theta\cos^2\theta+\sin\theta=\csc\theta\left(\cos^2\theta+\dfrac{\sin\theta}{\csc\theta}\right)$ Factoring

$=\csc\theta\left(\cos^2\theta+\dfrac{\sin\theta}{\frac{1}{\sin\theta}}\right)$ Reciprocal property

$=\csc\theta\left(\cos^2\theta+\sin^2\theta\right)$ Quotient property

$=\csc\theta(1)$ Pythagorean property

$=\csc\theta$

$\therefore\csc\theta\cos^2\theta+\sin\theta=\csc\theta$

2) $\csc\theta\cos^2\theta+\sin\theta=\dfrac{1}{\sin\theta}\cos^2\theta+\sin\theta$ Reciprocal property

$=\dfrac{\cos^2\theta}{\sin\theta}+\sin\theta\left(\dfrac{\sin\theta}{\sin\theta}\right)$ Multiplication property

$=\dfrac{\cos^2\theta+\sin^2\theta}{\sin\theta}$ Common denominator

$=\dfrac{1}{\sin\theta}$ Pythagorean property

$=\csc\theta$ Reciprocal property

$\therefore\csc\theta\cos^2\theta+\sin\theta=\csc\theta$

The following list contains techniques you can use to prove identities.

Steps in Proving Identities

1) Pick the side you wish to start with and write it down. Usually it is easier to start with the more complicated side.

2) Look for *algebraic* things to do.

   a) If there are two terms and you want only one,
      i) add fractions,
      ii) factor something out.

   b) Multiply by a clever form of 1
      i) to multiply a numerator or denominator by its conjugate,
      ii) to get a desired expression in the numerator or denominator.

   c) Do any obvious algebra such as distributing, squaring, or multiplying polynomials.

3) Look for *trigonometric* things to do.

   a) Look for familiar trigonometric expressions like
      $1-\cos^2 x$, $\cos x\sec x$, or $\dfrac{\sin x}{\cos x}$.

   b) If there are squares of functions, think of Pythagorean properties.

   c) Reduce the number of different functions, transforming them to the ones you want in the answer.

4) Keep looking at the answer to make sure you are headed in the right direction.

All material has been taken from *Trigonometry,* by P. Foerster, 3rd Edition