Section 3.3 – Properties Involving Functions of More Than One Argument

The fundamental properties (Reciprocal, Quotient, and Pythagorean) you learned in Sec 3.1 involve one argument only. In this section more than one argument appears, such as the sum of two numbers or the difference of two numbers. For example, you may have $\cos(\theta - PD)$, as when you graphed sinusoids in Chapter 2.

The operation cos does not distribute over addition and subtraction, so $\cos(\theta - PD) \neq \cos\theta - \cos PD$. In this section you will learn how to express functions of sums or differences of two angles in terms of functions of the angles themselves.

Objectives:
1) Be able to express functions of $-x$ in terms of functions of x.
2) Be able to express $\cos(A-B)$, $\cos(A+B)$, $\sin(A-B)$, and $\sin(A+B)$ in terms of $\sin A$, $\sin B$, $\cos A$, and $\cos B$.
3) Be able to express $\tan(A-B)$ and $\tan(A+B)$ in terms of $\tan A$ and $\tan B$.
4) Be able to solve equations and prove identities using these properties.

Before you get into the sum and difference properties, you need to take a look at even and odd properties and cofunction properties.

Even – Odd Properties

When applied to a function f, the words even and odd describe the symmetry that exists for the graph of the function.

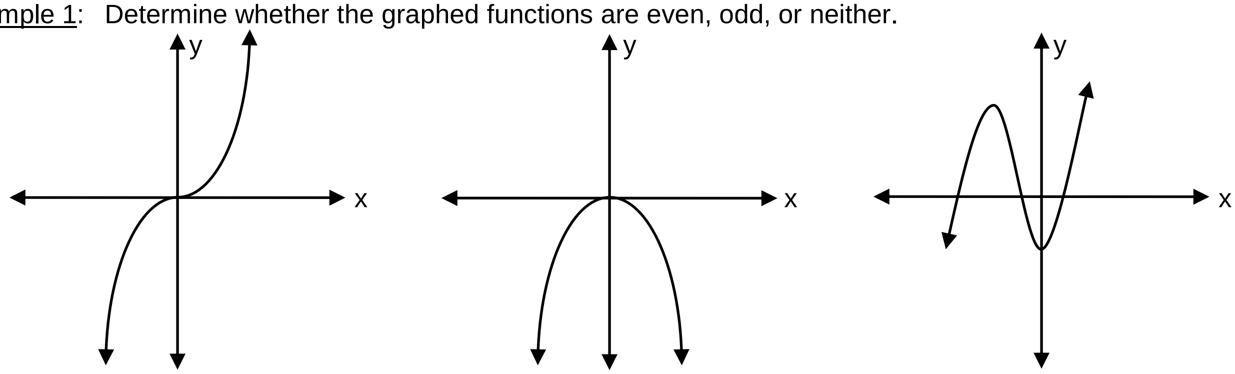
A function f is even if, for every number x in its domain, the number $-x$ is also in the domain and $f(-x) = f(x)$.

So, if f is even and (x, y) is on the graph of f, then the point $(-x, y)$ is also on the graph.
$\Rightarrow$ The graph is symmetric with respect to (wrt) the y-axis.

A function f is odd if, for every number x in its domain, the number $-x$ is also in the domain and $f(-x) = -f(x)$.

So, if f is odd and (x, y) is on the graph of f, then the point $(-x, -y)$ is also on the graph.
$\Rightarrow$ The graph is symmetric with respect to (wrt) the origin.

Example 1: Determine whether the graphed functions are even, odd, or neither.



Circle: Even, Odd, or Neither? Even, Odd, or Neither? Even, Odd, or Neither?

Example 2: Determine algebraically whether the functions are even, odd, or neither.

A) $f(x) = 4x^2 + 2$

$f(-x) = 4(-x)^2 + 2$

$= 4x^2 + 2$

$\Rightarrow f(-x) = f(x)$

$\Rightarrow$ Even function,

symmetric wrt the y-axis

B) $g(x) = 2x^3$

$g(-x) = 2(-x)^3$

$= -2x^3$

So, $g(-x) \neq g(x)$

$\Rightarrow$ Not an Even function

Now $-g(x) = -2x^3$

$\Rightarrow g(-x) = -g(x)$

$\Rightarrow$ Odd function, symmetric wrt the origin

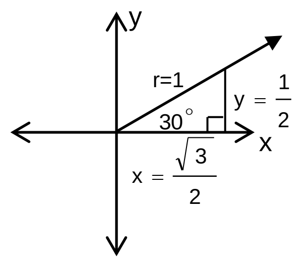
Section 3.3 – Properties Involving Functions of More Than One Argument (continued)

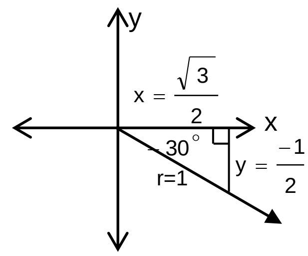
The trigonometric functions sine, cosecant, tangent, and cotangent are odd functions.
The functions cosine and secant are even functions.

Theorem: Even-Odd Properties

$\sin(-\theta) = -\sin\theta$	ODD	$\cos(-\theta) = \cos\theta$	EVEN	$\tan(-\theta) = -\tan\theta$	ODD
$\csc(-\theta) = -\csc\theta$	ODD	$\sec(-\theta) = \sec\theta$	EVEN	$\cot(-\theta) = -\cot\theta$	ODD

Example 3: Show that $\sin(-30^\circ) = -\sin 30^\circ$.





$$-30^\circ : (x, y) = \left(\frac{\sqrt{3}}{2}, \frac{-1}{2}\right) \qquad 30^\circ : (x, y) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

$$\sin(-30^\circ) = \frac{y}{r} = \frac{\frac{-1}{2}}{1} = \frac{-1}{2} \qquad -\sin(30^\circ) = \frac{-y}{r} = \frac{-\frac{1}{2}}{1} = \frac{-1}{2}$$

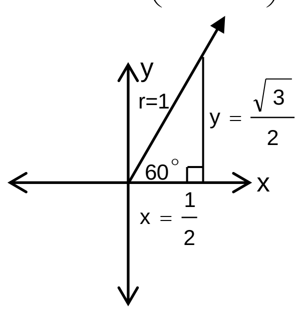
So, $\sin(-30^\circ) = -\sin 30^\circ$.

Example 4: Find the exact value of:

a) $\sin(-60^\circ)$

$\sin(-60^\circ) = -\sin 60^\circ$ since sine is odd and $60^\circ : (x, y) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$$= \frac{-y}{r} = \frac{-\sqrt{3}/2}{1} = \frac{-\sqrt{3}}{2}$$



b) $\sec(-30^\circ)$

$\sec(-30^\circ) = \sec 30^\circ$ since secant is even and $30^\circ : (x, y) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

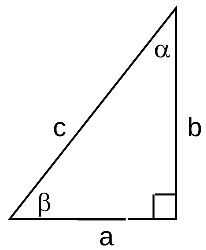
$$= \frac{r}{x} = \frac{1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3}}\right) = \frac{2\sqrt{3}}{3}$$

Section 3.3 – Properties Involving Functions of More Than One Argument (continued)

Complementary Angles: Cofunctions

Two acute angles are called complementary if their sum is 90°. The two acute angles of a right triangle are complementary.

Consider:



$$\sin\beta = \frac{\text{opp}}{\text{hyp}} = \frac{b}{c} \quad \text{and} \quad \cos\alpha = \frac{\text{adj}}{\text{hyp}} = \frac{b}{c}$$

So, $\frac{b}{c} = \frac{b}{c}$

$\Rightarrow \sin\beta = \cos\alpha$

Similarly: $\cos\beta = \frac{a}{c} = \sin\alpha$ $\quad \sec\beta = \frac{c}{a} = \csc\alpha$ $\quad \csc\beta = \frac{c}{b} = \sec\alpha$ $\quad \tan\beta = \frac{b}{a} = \cot\alpha$ $\quad \cot\beta = \frac{a}{b} = \tan\alpha$

Because of these relationships, the functions sine and cosine, tangent and cotangent, and secant and cosecant are called cofunctions of each other.

Complementary Angle Theorem: Cofunctions of complementary angles are equal.

So, if θ is an acute angle, the angle $90^\circ - \theta$ is the angle complementary to θ. Based on the Complementary Angle Theorem, the following are true:

$$\sin\theta = \cos(90^\circ - \theta) \qquad \csc\theta = \sec(90^\circ - \theta) \qquad \tan\theta = \cot(90^\circ - \theta)$$
$$\cos\theta = \sin(90^\circ - \theta) \qquad \sec\theta = \csc(90^\circ - \theta) \qquad \cot\theta = \tan(90^\circ - \theta)$$

And

$$\sin x = \cos\left(\frac{\pi}{2} - x\right) \qquad \csc x = \sec\left(\frac{\pi}{2} - x\right) \qquad \tan x = \cot\left(\frac{\pi}{2} - x\right)$$
$$\cos x = \sin\left(\frac{\pi}{2} - x\right) \qquad \sec x = \csc\left(\frac{\pi}{2} - x\right) \qquad \cot x = \tan\left(\frac{\pi}{2} - x\right)$$

Example 5: $\sin 30^\circ = \cos(90^\circ - 30^\circ)$ by the Complementary Angle Theorem

$= \cos(60^\circ)$

(30° and 60° are complementary angles)

Cofunctions

Example 6: $\csc\frac{\pi}{3} = \sec\left(\frac{\pi}{2} - \frac{\pi}{3}\right)$ by the Complementary Angle Theorem

$$= \sec\left(\frac{3\pi}{6} - \frac{2\pi}{6}\right)$$

$$= \sec\left(\frac{\pi}{6}\right)$$

Section 3.3 – Properties Involving Functions of More Than One Argument (continued)

Example 7: Find the exact value of

a) $\sin 38^\circ - \cos 52^\circ$

$$\sin 38^\circ - \cos 52^\circ = \sin 38^\circ - \sin(90^\circ - 52^\circ) \text{ by the Complementary Angle Theorem}$$
$$= \sin 38^\circ - \sin 38^\circ$$
$$= 0$$

OR

$$\sin 38^\circ - \cos 52^\circ = \cos(90^\circ - 38^\circ) - \cos 52^\circ \text{ by the Complementary Angle Theorem}$$
$$= \cos 52^\circ - \cos 52^\circ$$
$$= 0$$

b) $\dfrac{\cos 10^\circ}{\sin 80^\circ}$

$$\frac{\cos 10^\circ}{\sin 80^\circ} = \frac{\sin(90^\circ - 10^\circ)}{\sin 80^\circ} \text{ by the Complementary Angle Theorem}$$
$$= \frac{\sin 80^\circ}{\sin 80^\circ}$$
$$= 1$$

OR

$$\frac{\cos 10^\circ}{\sin 80^\circ} = \frac{\cos 10^\circ}{\cos(90^\circ - 80^\circ)} \text{ by the Complementary Angle Theorem}$$
$$= \frac{\cos 10^\circ}{\cos 10^\circ}$$
$$= 1$$

Sum and Difference Formulas

Formulas that involve the sum or difference of two angles are called the sum and difference formulas.

Theorem: Sum and Difference Formulas for Cosines

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$
$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

Example 8: Find the exact value of $\cos\left(\dfrac{7\pi}{12}\right)$.

$$\cos\left(\frac{7\pi}{12}\right) = \cos\left(\frac{4\pi}{12} + \frac{3\pi}{12}\right)$$
$$= \cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$$
$$= \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) \quad \text{by the theorem above}$$
$$= \left(\frac{x}{r}\right)\left(\frac{x}{r}\right) - \left(\frac{y}{r}\right)\left(\frac{y}{r}\right) \quad \text{Now } \frac{\pi}{3} : (x, y) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \text{ and } \frac{\pi}{4} : (x, y) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right).$$

Section 3.3 – Properties Involving Functions of More Than One Argument (continued)

$$= \left(\frac{\frac{1}{2}}{1} \right) \left(\frac{\frac{\sqrt{2}}{2}}{1} \right) - \left(\frac{\frac{\sqrt{3}}{2}}{1} \right) \left(\frac{\frac{\sqrt{2}}{2}}{1} \right)$$

$$= \left(\frac{1}{2} \right) \left(\frac{\sqrt{2}}{2} \right) - \left(\frac{\sqrt{3}}{2} \right) \left(\frac{\sqrt{2}}{2} \right)$$

$$= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}$$

$$= \frac{1}{4} \left(\sqrt{2} - \sqrt{6} \right)$$

Theorem: Sum and Difference Formulas for Sines

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

Example 9: Find the exact value of $\sin\left(\frac{5\pi}{12} \right)$.

$$\sin\left(\frac{5\pi}{12} \right) = \sin\left(\frac{3\pi}{12} + \frac{2\pi}{12} \right)$$

$$= \sin\left(\frac{\pi}{4} + \frac{\pi}{6} \right)$$

$$= \sin\left(\frac{\pi}{4} \right)\cos\left(\frac{\pi}{6} \right) + \cos\left(\frac{\pi}{4} \right)\sin\left(\frac{\pi}{6} \right)$$ by the theorem above

$$= \left(\frac{y}{r} \right)\left(\frac{x}{r} \right) + \left(\frac{x}{r} \right)\left(\frac{y}{r} \right)$$ Now $\frac{\pi}{4} : (x, y) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right)$ and $\frac{\pi}{6} : (x, y) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2} \right)$.

$$= \left(\frac{\frac{\sqrt{2}}{2}}{1} \right) \left(\frac{\frac{\sqrt{3}}{2}}{1} \right) + \left(\frac{\frac{\sqrt{2}}{2}}{1} \right) \left(\frac{\frac{1}{2}}{1} \right)$$

$$= \left(\frac{\sqrt{2}}{2} \right) \left(\frac{\sqrt{3}}{2} \right) + \left(\frac{\sqrt{2}}{2} \right) \left(\frac{1}{2} \right)$$

$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$

$$= \frac{1}{4} \left(\sqrt{6} + \sqrt{2} \right)$$

Example 10: Find the exact value of $\cos 80^\circ \cos 20^\circ + \sin 80^\circ \sin 20^\circ$.

$$\cos 80^\circ \cos 20^\circ + \sin 80^\circ \sin 20^\circ = \cos(80^\circ - 20^\circ)$$ by the Sum and Difference Theorem for Cosines

$$= \cos 60^\circ$$

$$= \frac{x}{r}$$ Now $60^\circ : (x, y) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right)$.

$$= \frac{\frac{1}{2}}{1}$$

$$= \frac{1}{2}$$

Section 3.3 – Properties Involving Functions of More Than One Argument (continued)

Example 11: Given $\sin A = \dfrac{3}{5},\ 0 < A < \dfrac{\pi}{2};\ \cos B = \dfrac{2\sqrt{5}}{5},\ \dfrac{-\pi}{2} < B < 0.$

Preliminary Work:

A is in quadrant I $\Rightarrow$ $\cos A$ is positive

$$\cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \sqrt{1 - \left(\frac{9}{25}\right)} = \sqrt{\frac{25}{25} - \frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

B is in quadrant IV $\Rightarrow$ $\sin B$ is negative

$$\sin B = -\sqrt{1 - \cos^2 B} = -\sqrt{1 - \left(\frac{2\sqrt{5}}{5}\right)^2} = -\sqrt{1 - \left(\frac{4}{5}\right)} = -\sqrt{\frac{5}{5} - \frac{4}{5}} = -\sqrt{\frac{1}{5}} = \frac{-1}{\sqrt{5}} = \frac{-1}{\sqrt{5}}\left(\frac{\sqrt{5}}{\sqrt{5}}\right) = \frac{-\sqrt{5}}{5}$$

a) $$\sin(A + B) = \sin A \cos B + \cos A \sin B = \left(\frac{3}{5}\right)\left(\frac{2\sqrt{5}}{5}\right) + \left(\frac{4}{5}\right)\left(\frac{-\sqrt{5}}{5}\right) = \frac{6\sqrt{5}}{25} - \frac{4\sqrt{5}}{25} = \frac{2\sqrt{5}}{25}$$

b) $$\sin(A - B) = \sin A \cos B - \cos A \sin B = \left(\frac{3}{5}\right)\left(\frac{2\sqrt{5}}{5}\right) - \left(\frac{4}{5}\right)\left(\frac{-\sqrt{5}}{5}\right) = \frac{6\sqrt{5}}{25} + \frac{4\sqrt{5}}{25} = \frac{10\sqrt{5}}{25} = \frac{2\sqrt{5}}{5}$$

c) $$\cos(A + B) = \cos A \cos B - \sin A \sin B = \left(\frac{4}{5}\right)\left(\frac{2\sqrt{5}}{5}\right) - \left(\frac{3}{5}\right)\left(\frac{-\sqrt{5}}{5}\right) = \frac{8\sqrt{5}}{25} + \frac{3\sqrt{5}}{25} = \frac{11\sqrt{5}}{25}$$

Section 3.3 – Properties Involving Functions of More Than One Argument (continued)

Theorem: Sum and Difference Formulas for Tangents

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \qquad \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Example 12: Find the exact value of $\tan 15^\circ$.

$$\tan 15^\circ = \tan(60^\circ - 45^\circ)$$

$$= \frac{\tan 60^\circ - \tan 45^\circ}{1 + \tan 60^\circ \tan 45^\circ}$$

Now $60^\circ : (x_1, y_1) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $45^\circ : (x_2, y_2) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$$= \frac{\frac{y_1}{x_1} - \frac{y_2}{x_2}}{1 + \left(\frac{y_1}{x_1}\right)\left(\frac{y_2}{x_2}\right)}$$

$$= \frac{\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right) - \left(\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}\right)}{1 + \left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right)\left(\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}\right)}$$

$$= \frac{\sqrt{3} - 1}{1 + \sqrt{3}(1)}$$

$$= \frac{\sqrt{3} - 1}{1 + \sqrt{3}}$$

$$= \frac{\sqrt{3} - 1}{1 + \sqrt{3}}\left(\frac{1 - \sqrt{3}}{1 - \sqrt{3}}\right)$$

$$= \frac{\sqrt{3} - 3 - 1 + \sqrt{3}}{1 - \cancel{\sqrt{3}} + \cancel{\sqrt{3}} - 3}$$

$$= \frac{2\sqrt{3} - 4}{-2}$$

$$= \frac{\cancel{-2}(2 - \sqrt{3})}{\cancel{-2}}$$

$$= 2 - \sqrt{3}$$

All material has been taken from *Trigonometry,* by P. Foerster, 3rd Edition