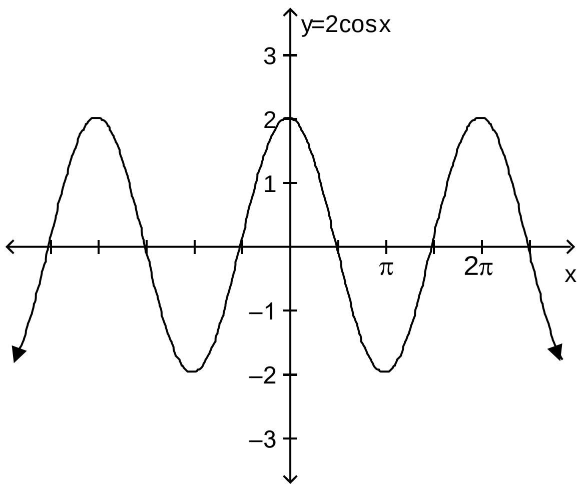
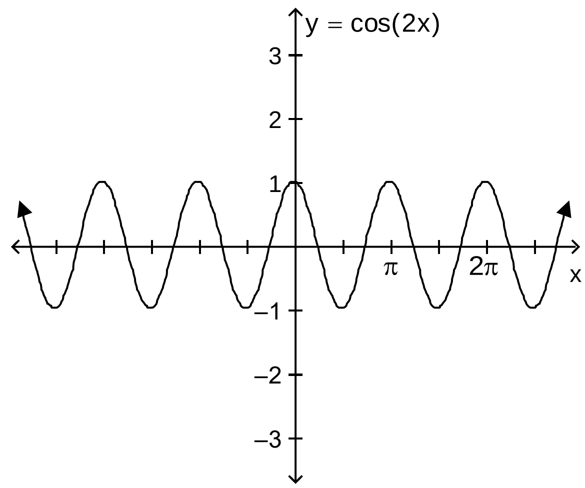
Section 3.4 – Multiple Argument Properties

In algebra, you learned that you can commute factors in a product. For example, $xyz = zxy$. The same is not true for trigonometric operations. For example, $\cos(2x) \neq 2\cos x$. In $\cos(2x)$, we know the 2 affects the period, whereas in $2\cos x$, the 2 affects the amplitude. You can also see that the expressions are not equivalent by looking at their graphs.





There are properties that relate trigonometric functions of twice an argument to functions of that argument alone. The composite argument properties can be used to derive these properties. The multiple argument properties relate functions of "double angles" to functions of the "single angle."

Objective: Be able to express $\sin(2A)$, $\cos(2A)$, and $\tan(2A)$ in terms of $\sin A$, $\cos A$, and $\tan A$.

Theorem: Double-Angle Formulas (Double Argument Properties)

$$\sin(2A) = 2\sin A\cos A \qquad \cos(2A) = \cos^2 A - \sin^2 A \qquad \tan(2A) = \frac{2\tan A}{1-\tan^2 A}$$

$$\cos(2A) = 1 - 2\sin^2 A$$

$$\cos(2A) = 2\cos^2 A - 1$$

Proof: $\sin(2A) = \sin(A + A)$

$= \sin A\cos A + \cos A\sin A$ by the Sum Formula Theorem for Sines

$= 2\sin A\cos A$

$\therefore \sin(2A) = 2\sin A\cos A$

$\cos(2A) = \cos(A + A)$

$= \cos A\cos A - \sin A\sin A$ by the Sum Formula Theorem for Cosines

$= \cos^2 A - \sin^2 A$

$\therefore \cos(2A) = \cos^2 A - \sin^2 A$

Now $\cos(2A) = \cos^2 A - \sin^2 A$

$= \left(1 - \sin^2 A\right) - \sin^2 A$ by the Pythagorean Property

$= 1 - 2\sin^2 A$

$\therefore \cos(2A) = 1 - 2\sin^2 A$

Now $\cos(2A) = \cos^2 A - \sin^2 A$

$= \cos^2 A - \left(1 - \cos^2 A\right)$ by the Pythagorean Property

$= 2\cos^2 A - 1$

$\therefore \cos(2A) = 2\cos^2 A - 1$

Section 3.4 – Multiple Argument Properties (continued)

Now $\tan(2A) = \tan(A + A)$

$$= \frac{\tan A + \tan A}{1 - \tan A \tan A}$$

$$= \frac{2\tan A}{1 - \tan^2 A}$$

You can also find $\tan(2A)$ by setting $\tan(2A) = \dfrac{\sin(2A)}{\cos(2A)}$.

Example 1: Given $A = 60^\circ$, use the double angle (argument) formulas to find $\sin(2A)$, $\cos(2A)$, and $\tan(2A)$.

$$\sin(2A) = 2\sin A \cos A$$

$$\sin\left[2(60^\circ)\right] = 2\sin 60^\circ \cos 60^\circ$$

$$= 2\left(\frac{y}{r}\right)\left(\frac{x}{r}\right) \qquad 60^\circ : (x, y) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

$$= 2\left(\frac{\frac{\sqrt{3}}{2}}{1}\right)\left(\frac{\frac{1}{2}}{1}\right)$$

$$= 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)$$

$$= 2\left(\frac{\sqrt{3}}{4}\right)$$

$$= \frac{\sqrt{3}}{2}$$

$$\cos(2A) = \cos^2 A - \sin^2 A$$

$$\cos\left[2(60^\circ)\right] = \cos^2 60^\circ - \sin^2 60^\circ$$

$$= \left(\frac{x}{r}\right)^2 - \left(\frac{y}{r}\right)^2$$

$$= \left(\frac{\frac{1}{2}}{1}\right)^2 - \left(\frac{\frac{\sqrt{3}}{2}}{1}\right)^2$$

$$= \left(\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2$$

$$= \frac{1}{4} - \frac{3}{4}$$

$$= \frac{-2}{4}$$

$$= \frac{-1}{2}$$

OR

$$\cos(2A) = 1 - 2\sin^2 A$$

$$\cos\left[2(60^\circ)\right] = 1 - 2\sin^2 60^\circ$$

$$= 1 - 2\left(\frac{y}{r}\right)^2$$

$$= 1 - 2\left(\frac{\frac{\sqrt{3}}{2}}{1}\right)^2$$

$$= 1 - 2\left(\frac{\sqrt{3}}{2}\right)^2$$

$$= 1 - 2\left(\frac{3}{4}\right)$$

$$= \frac{4}{4} - \frac{6}{4}$$

$$= \frac{-2}{4}$$

$$= \frac{-1}{2}$$

OR

$$\cos(2A) = 2\cos^2 A - 1$$

$$\cos\left[2(60^\circ)\right] = 2\cos^2 60^\circ - 1$$

$$= 2\left(\frac{x}{r}\right)^2 - 1$$

$$= 2\left(\frac{\frac{1}{2}}{1}\right)^2 - 1$$

$$= 2\left(\frac{1}{2}\right)^2 - 1$$

$$= 2\left(\frac{1}{4}\right) - 1$$

$$= \frac{1}{2} - 1$$

$$= \frac{-1}{2}$$

Section 3.4 – Multiple Argument Properties (continued)

Example 1 Continued:

$$\tan(2A) = \frac{2\tan A}{1-\tan^2 A}$$

$$\tan\left[2(60^\circ)\right] = \frac{2\tan 60^\circ}{1-\tan^2 60^\circ}$$

$$= \frac{2\left(\frac{y}{x}\right)}{1-\left(\frac{y}{x}\right)^2}$$

$$= \frac{2\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right)}{1-\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right)^2}$$

$$= \frac{2\sqrt{3}}{1-\left(\sqrt{3}\right)^2}$$

$$= \frac{2\sqrt{3}}{1-3}$$

$$= \frac{2\sqrt{3}}{-2}$$

$$= -\sqrt{3}$$

Example 2: Given $\sin A = \frac{-4}{5}$ and A terminates in Quadrant III, use the double angle (argument) formulas to find $\sin(2A)$, $\cos(2A)$, and $\tan(2A)$.

Use Pythagorean identities and the given value of $\sin A$ to find $\cos A$ and $\tan A$.
Angle A is in Quadrant III, so $\cos A$ is negative and $\tan A$ is positive.

$$\sin^2 A + \cos^2 A = 1$$

$$\cos A = -\sqrt{1-\sin^2 A} \quad \text{Quad III, cos neg}$$

$$= -\sqrt{1-\left(\frac{-4}{5}\right)^2}$$

$$= -\sqrt{1-\left(\frac{16}{25}\right)}$$

$$= -\sqrt{\frac{9}{25}}$$

$$= \frac{-3}{5}$$

$$1+\tan^2 A = \sec^2 A$$

$$\tan A = \sqrt{\sec^2 A - 1} \quad \text{Quad III, tan pos}$$

$$= \sqrt{\frac{1}{\cos^2 A} - 1}$$

$$= \sqrt{\frac{1}{\left(\frac{-3}{5}\right)^2} - 1}$$

$$= \sqrt{\frac{1}{\left(\frac{9}{25}\right)} - 1}$$

$$= \sqrt{\frac{25}{9} - \frac{9}{9}}$$

$$= \sqrt{\frac{16}{9}}$$

$$= \frac{4}{3}$$

So, $\cos A = \frac{-3}{5}$ and $\tan A = \frac{4}{3}$.

Section 3.4 – Multiple Argument Properties (continued)

Example 2 Continued:

$$\sin(2A) = 2\sin A\cos A = 2\left(\frac{-4}{5}\right)\left(\frac{-3}{5}\right) = 2\left(\frac{12}{25}\right) = \frac{24}{25} = 0.96$$

$$\cos(2A) = \cos^2 A - \sin^2 A = \left(\frac{-3}{5}\right)^2 - \left(\frac{-4}{5}\right)^2 = \left(\frac{9}{25}\right) - \left(\frac{16}{25}\right) = \frac{-7}{25} = -0.28$$

$$\tan A = \frac{2\tan A}{1-\tan^2 A} = \frac{2\left(\frac{4}{3}\right)}{1-\left(\frac{4}{3}\right)^2} = \frac{\frac{8}{3}}{1-\frac{16}{9}} = \frac{\frac{8}{3}}{\frac{-7}{9}} = \frac{8}{3}\left(\frac{9}{-7}\right) = \frac{72}{-21} = \frac{-24}{7} \approx -3.43$$

Checks:

$$A = \sin^{-1}\left(\frac{-4}{5}\right)$$
$$A = -53.13^\circ$$
$$A_{ref} = 53.13^\circ$$

A in Quad III, so

$$A = 180^\circ + A_{ref} = 180^\circ + 53.13^\circ = 233.13^\circ$$
$$2A = 466.26^\circ$$

$$\sin(2A) = \sin 466.26^\circ \approx 0.96$$

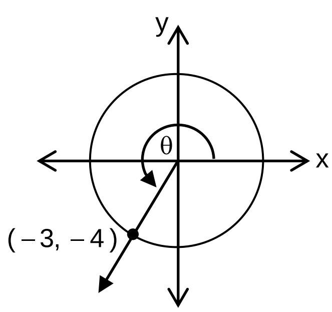
$$\cos(2A) = \cos 466.26^\circ \approx -0.28$$

$$\tan(2A) = \tan 466.26^\circ \approx -3.43$$

Example 3: Given $\tan\theta = \frac{4}{3},\ \pi < \theta < \frac{3\pi}{2}$.

Find the exact value of a) $\sin(2\theta)$ b) $\cos(2\theta)$

Now $\pi < \theta < \frac{3\pi}{2} \Rightarrow \theta$ in quadrant III, so x and y are negative and $\sin\theta$ and $\cos\theta$ are negative.



$$\tan\theta = \frac{4}{3}$$
$$= \frac{y}{x} \Rightarrow \text{Assume } x = -3,\ y = -4$$
$$x^2 + y^2 = r^2$$
$$(-3)^2 + (-4)^2 = r^2$$
$$9 + 16 = r^2$$
$$25 = r^2$$
$$r = \pm\sqrt{25}$$
$$= \pm 5$$
$= 5$ r represents hypotenuse, positive

a) $$\sin(2\theta) = 2\sin\theta\cos\theta = 2\left(\frac{y}{r}\right)\left(\frac{x}{r}\right) = 2\left(\frac{-4}{5}\right)\left(\frac{-3}{5}\right) = 2\left(\frac{12}{25}\right) = \frac{24}{25}$$

b) $$\cos(2\theta) = \cos^2\theta - \sin^2\theta = \left(\frac{x}{r}\right)^2 - \left(\frac{y}{r}\right)^2 = \left(\frac{-3}{5}\right)^2 - \left(\frac{-4}{5}\right)^2 = \frac{9}{25} - \frac{16}{25} = \frac{-7}{25}$$

Section 3.4 – Multiple Argument Properties (continued)

Example 4: Find the exact values for $\sin(2A)$, $\cos(2A)$, and $\tan(2A)$ if A terminates in the given quadrant.

Given: $\sin A = \frac{-2}{7}$, Quadrant IV

$\cos A = \sqrt{1-\sin^2 A}$ Quadrant IV $\Rightarrow$ cos positive

$$\cos A = \sqrt{1-\left(\frac{-2}{7}\right)^2} = \sqrt{1-\frac{4}{49}} = \sqrt{\frac{45}{49}} = \frac{\sqrt{9}\sqrt{5}}{7} = \frac{3\sqrt{5}}{7}$$

So, $\sin(2A) = 2\sin A\cos A$

$$= 2\left(\frac{-2}{7}\right)\left(\frac{3\sqrt{5}}{7}\right) = 2\left(\frac{-6\sqrt{5}}{49}\right) = \frac{-12\sqrt{5}}{49}$$

$\cos(2A) = 1-2\sin^2 A$

$$= 1-2\left(\frac{-2}{7}\right)^2 = 1-2\left(\frac{4}{49}\right) = 1-\frac{8}{49} = \frac{41}{49}$$

$\tan(2A) = \frac{\sin(2A)}{\cos(2A)}$

$$= \frac{\frac{-12\sqrt{5}}{49}}{\frac{41}{49}} = \frac{-12\sqrt{5}}{41}$$

You might be asked to rewrite $\cos(2\theta)$ in terms of $\sin\theta$.

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$= \left(1 - \sin^2\theta\right) - \sin^2\theta \quad \text{by the Pythagorean Property}$$
$$= 1 - 2\sin^2\theta$$
$$\therefore \cos(2\theta) = 1 - 2\sin^2\theta$$

Example 5: Write $\tan(14x)$ in terms of $\tan(7x)$.

$$\tan(14x) = \tan(7x+7x) = \frac{\tan(7x)+\tan(7x)}{1-\tan(7x)\tan(7x)} = \frac{2\tan(7x)}{1-\tan^2(7x)}$$

Example 6: Write $\cos x$ in terms of $\cos\left(\frac{1}{2}x\right)$ alone.

$$\cos x = \cos\left[2\left(\frac{1}{2}x\right)\right] = 2\cos^2\left(\frac{1}{2}x\right)-1$$

since $\cos(2A) = 2\cos^2 A - 1$

All material has been taken from *Trigonometry,* by P. Foerster, 3rd Edition