Section 5.2 – Oblique Triangles – Law of Cosines

Objectives:
1) Given two sides and the included angle, find the length of the third side of the triangle.
2) Given three sides of a triangle, find the measure of a specified angle.

In Section 5.1, we found unknown measures in right triangles. Here we learn how to do the same things for oblique triangles, which do not have a right angle.

To solve a triangle means to find the missing lengths of its sides and the measurements of its angles. We will express the lengths of the sides rounded to two decimal places and the angles in degrees rounded to one decimal place.

Theorem: Law of Cosines

For a triangle with sides a, b, c and opposite angles A, B, C, respectively,

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

The Law of Cosines says the following:

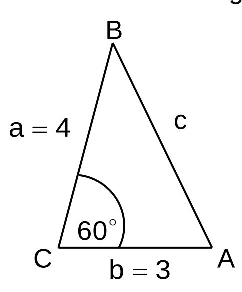
The square of one side of a triangle equals the sum of the squares of the other two sides minus twice their product times the cosine of their included angle.

Note that if the included angle is 90°, the triangle is a right triangle and the formula above becomes the Pythagorean Theorem (say $C = 90^\circ$, then $c^2 = a^2 + b^2$). Thus, the Pythagorean Theorem is a special case of the Law of Cosines.

The Law of Cosines is used to solve two types of triangles, **SAS** and **SSS**. Remember from geometry, that SAS (side-angle-side) means two sides and the included angle are known. SSS (side-side-side) means three sides are known.

Example 1: SAS Case:

Solve the triangle: $a = 4$, $b = 3$, $m\angle C = 60^\circ$.



Find side c using the Law of Cosines.

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos C \\ &= 4^2 + 3^2 - 2(4)(3)\cos 60^\circ \\ &= 16 + 9 - 8(3)\cos 60^\circ \\ &= 25 - 24\cos 60^\circ \\ &= 25 - 24\left(\frac{x}{r}\right) \qquad 60^\circ : (x, y) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \\ &= 25 - 24\left(\frac{1/2}{1}\right) \\ &= 25 - 24\left(\frac{1}{2}\right) \\ &= 25 - 12 \\ c^2 &= 13 \end{aligned}$$

So, $c = \sqrt{13}$

$c \approx 3.605$

$c \approx 3.61$

Now use the Law of Cosines, and this rounded value of c, to find the measures of angles A and B.

Section 5.2 – Oblique Triangles – Law of Cosines (continued)

Example 1 (continued):

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$2bc\cos A = b^2 + c^2 - a^2$$
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$
$$= \frac{3^2 + (3.61)^2 - 4^2}{2(3)(3.61)}$$
$$= \frac{9 + 13.0361 - 16}{6(3.61)}$$
$$= \frac{6.0321}{21.66}$$
$$\cos A \approx 0.278490$$
So, $m\angle A = \cos^{-1}(0.278490)$
$$m\angle A \approx 73.82^\circ$$
$$m\angle A \approx 73.8^\circ$$

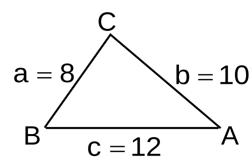
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$2ac\cos B = a^2 + c^2 - b^2$$
$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$
$$= \frac{4^2 + (3.61)^2 - 3^2}{2(4)(3.61)}$$
$$= \frac{16 + 13.0321 - 9}{8(3.61)}$$
$$= \frac{20.0321}{28.88}$$
$$\cos B \approx 0.693632$$
So, $m\angle B = \cos^{-1}(0.693632)$
$$m\angle B \approx 46.08^\circ$$
$$m\angle B \approx 46.1^\circ$$

Check: $m\angle A + m\angle B + m\angle C = 73.8^\circ + 46.1^\circ + 60^\circ$

$= 179.9^\circ$ (Close enough to 180°: not exact due to rounding errors)

Therefore, $m\angle A = 73.8^\circ$, $m\angle B = 46.1^\circ$, and $c = 3.61$.

Example 2: SSS Case:
Solve the triangle: $a = 8$, $b = 10$, $c = 12$.



To find A:

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$2bc\cos A = b^2 + c^2 - a^2$$
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$
$$= \frac{10^2 + 12^2 - 8^2}{2(10)(12)}$$
$$= \frac{100 + 144 - 64}{20(12)}$$
$$= \frac{180}{240}$$
$$\cos A = \frac{3}{4}$$
$$\cos A = 0.75$$
So, $m\angle A = \cos^{-1}(0.75)$
$$m\angle A \approx 41.40^\circ$$
$$m\angle A \approx 41.4^\circ$$

To find B:

$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$2ac\cos B = a^2 + c^2 - b^2$$
$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$
$$= \frac{8^2 + 12^2 - 10^2}{2(8)(12)}$$
$$= \frac{64 + 144 - 100}{16(12)}$$
$$= \frac{108}{192}$$
$$\cos B = \frac{9}{16}$$
$$\cos B = 0.5625$$
So, $m\angle B = \cos^{-1}(0.5625)$
$$m\angle B \approx 55.77^\circ$$
$$m\angle B \approx 55.8^\circ$$

To find C:

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$2ab\cos C = a^2 + b^2 - c^2$$
$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$
$$= \frac{8^2 + 10^2 - 12^2}{2(8)(10)}$$
$$= \frac{64 + 100 - 144}{16(10)}$$
$$= \frac{20}{160}$$
$$\cos C = \frac{1}{8}$$
$$\cos C = 0.125$$
So, $m\angle C = \cos^{-1}(0.125)$
$$m\angle C \approx 82.81^\circ$$
$$m\angle C \approx 82.8^\circ$$

Check: $m\angle A + m\angle B + m\angle C = 41.4^\circ + 55.8^\circ + 82.8^\circ$

$= 180^\circ$

(Once you have found two angles, you could use $m\angle C = 180^\circ - m\angle A - m\angle B = 180^\circ - 41.4^\circ - 55.8^\circ = 82.8^\circ$.
But, **finding all three angles** is a better way to check your work, so it **will be expected in this class**.)

All material has been taken from *Trigonometry,* by P. Foerster, 3rd Edition