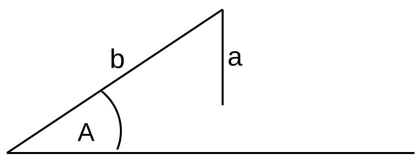
Section 5.5 – The Ambiguous Case

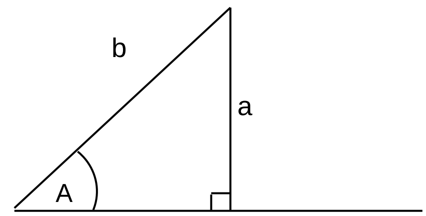
Objective: Given SSA, determine whether or not there are possible triangles, and if so, find the other side length and angle measures.

You may recall from geometry that when given two sides and a non-included angle, the SSA case, you may not necessarily be able to construct a triangle. There are four ways a triangle ABC would come out if you knew the lengths of a and b and the measure of angle A. To see this, consider the following figures.

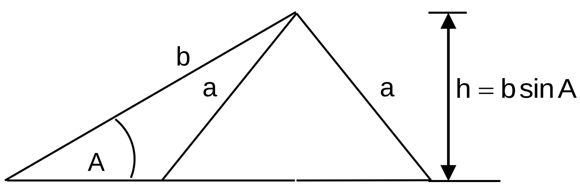
No triangle: If side a is not long enough to form a triangle.



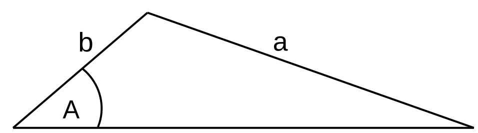
One right triangle: If side a is just long enough to form a right triangle.



Two triangles: If $a < b$ and $h(= b\sin A) < a$, then two distinct triangles can be formed.



One triangle: If $a \geq b$, then only one triangle can be formed.

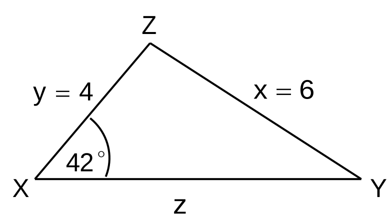


So, when given SSA information for a triangle, one, two, or no triangles are possible. For this reason, SSA is called the ambiguous case. Ambiguous means "two or more possible meanings."

Recall that if you were given two angles, SAA or ASA, you used the Law of Sines to solve the triangle. If you were given at least two sides, SAS or SSS, you used the Law of Cosines to solve the triangle. To solve a **SSA** triangle, again given two sides, you may use the Law of Cosines and the quadratic formula to determine how many triangles are possible and then find the missing side and angles for each possible triangle.

Example 1: Case: SSA

Solve ΔXYZ, given: $x = 6$, $y = 4$, $m\angle X = 42^\circ$.



$x^2 = y^2 + z^2 - 2yz\cos X$ Law of Cosines

$6^2 = 4^2 + z^2 - 2(4)z\cos 42^\circ$

$36 = 16 + z^2 - 8z\cos 42^\circ$

$0 = -20 + z^2 - 8z\cos 42^\circ$

$0 = z^2 - 8\cos 42^\circ z - 20$ A quadratic equation in the variable z

$0 = z^2 - 5.945159z - 20$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$z = \frac{-(-5.945159) \pm \sqrt{(-5.945159)^2 - 4(1)(-20)}}{2(1)}$$

$$z = \frac{5.945159 \pm \sqrt{35.344916 + 80}}{2}$$

Section 5.5 – The Ambiguous Case (continued)

Example 1 (continued):

$$z = \frac{5.945159 \pm \sqrt{115.344916}}{2}$$

$$z = \frac{5.945159 \pm 10.739875}{2}$$

$$z = \frac{5.945159 + 10.739875}{2} \quad \text{or} \quad z = \frac{5.945159 - 10.739875}{2}$$

$$z = \frac{16.685034}{2} \quad \text{or} \quad z = \frac{-4.794716}{2}$$

$$z = 8.342517 \quad \text{or} \quad z = -2.397358$$

$$z \approx 8.34 \quad \text{or} \quad z \approx -2.40$$

Side length z cannot be negative, so there is only one triangle with side length $z = 8.34$.
One triangle with: $x = 6$, $y = 4$, $z = 8.34$, $m\angle X = 42^\circ$.

You can use the Law of Cosines to find the remaining two angles of the triangle.

$$y^2 = x^2 + z^2 - 2xz\cos Y$$
$$2xz\cos Y = x^2 + z^2 - y^2$$
$$\cos Y = \frac{x^2 + z^2 - y^2}{2xz}$$
$$= \frac{6^2 + (8.34)^2 - 4^2}{2(6)(8.34)}$$
$$= \frac{36 + 69.5556 - 16}{12(8.34)}$$
$$= \frac{89.5556}{100.08}$$
$$\cos Y \approx 0.894840$$
So, $m\angle Y = \cos^{-1}(0.894840)$
$$m\angle Y \approx 26.51^\circ$$
$$m\angle Y \approx 26.5^\circ$$

$$z^2 = x^2 + y^2 - 2xy\cos Z$$
$$2xy\cos Z = x^2 + y^2 - z^2$$
$$\cos Z = \frac{x^2 + y^2 - z^2}{2xy}$$
$$= \frac{6^2 + 4^2 - (8.34)^2}{2(6)(4)}$$
$$= \frac{36 + 16 - 69.5556}{12(4)}$$
$$= \frac{-17.5556}{48}$$
$$\cos Z \approx -0.365742$$
So, $m\angle Z = \cos^{-1}(-0.365742)$
$$m\angle Z \approx 111.45^\circ$$
$$m\angle Z \approx 111.5^\circ$$

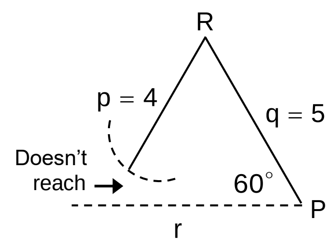
Check: $m\angle X + m\angle Y + m\angle Z = 42^\circ + 26.5^\circ + 111.5^\circ$
$$= 180^\circ$$

Thus, you have one triangle with the following:
$x = 6$, $y = 4$, $z = 8.34$, $m\angle X = 42^\circ$, $m\angle Y = 26.5^\circ$, and $m\angle Z = 111.5^\circ$

Section 5.5 – The Ambiguous Case (continued)

Example 2: Case: SSA
Find the possible lengths of the indicated side.
In ΔPQR, $p = 4$, $q = 5$, $m\angle P = 60^\circ$. Find r.

$$p^2 = q^2 + r^2 - 2qr\cos P$$
$$4^2 = 5^2 + r^2 - 2(5)r\cos 60^\circ$$
$$16 = 25 + r^2 - 10\cos 60^\circ r$$
$$0 = 9 + r^2 - 10\left(\frac{1}{2}\right)r$$
$$0 = r^2 - 5r + 9$$ A quadratic equation in the variable r
$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$r = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(9)}}{2(1)}$$
$$r = \frac{5 \pm \sqrt{25 - 36}}{2}$$
$$r = \frac{5 \pm \sqrt{-11}}{2}$$

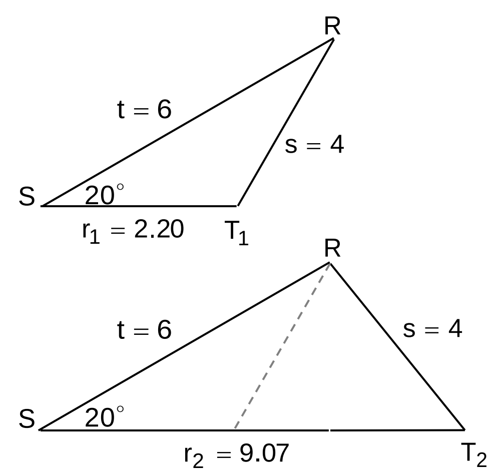


You cannot take the square root of a negative number, so there is no real solution. Thus, there is no r value and, therefore, there is no triangle.

Example 3: Case: SSA
Find the possible lengths of the indicated side.
In ΔRST, $s = 4$, $t = 6$, $m\angle S = 20^\circ$. Find r.

$$s^2 = r^2 + t^2 - 2rt\cos S$$
$$4^2 = r^2 + 6^2 - 2r(6)\cos 20^\circ$$
$$16 = r^2 + 36 - 12r\cos 20^\circ$$
$$0 = r^2 - 12\cos 20^\circ r + 20$$
$$0 = r^2 - 11.276311r + 20$$ A quadratic equation in the variable r
$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$r = \frac{-(-11.276311) \pm \sqrt{(-11.276311)^2 - 4(1)(20)}}{2(1)}$$
$$r = \frac{11.276311 \pm \sqrt{127.155190 - 80}}{2}$$
$$r = \frac{11.276311 \pm \sqrt{47.155190}}{2}$$
$$r = \frac{11.276311 \pm 6.866964}{2}$$
$$r = \frac{11.276311 + 6.866964}{2} \text{ or } r = \frac{11.276311 - 6.866964}{2}$$
$$r = \frac{18.143275}{2} \text{ or } r = \frac{4.409347}{2}$$
$$r = 9.0716375 \text{ or } r = 2.2046735$$

So, $r \approx 2.20$ or $r \approx 9.07$.
Thus, there are two triangles.



All material has been taken from *Trigonometry,* by P. Foerster, 3rd Edition