Completing the Square and Circles, Ellipses, and Hyperbolas

It is often helpful to write a quadratic expression as a perfect square. A method called completing the square may be used to make any quadratic expression a perfect square.

Completing the Square: To complete the square of the expression $x^2 + bx$, add the square of half of the coefficient of x, that is, add $\left(\frac{b}{2}\right)^2$.

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

Example 1: Find the term that should be added to the expression to create a perfect square trinomial (perfect square).

a) $x^2 - 8x$

$x^2 + -8x$

$\left(\frac{b}{2}\right)^2 = \left(\frac{-8}{2}\right)^2$

$= (-4)^2$

$= 16$

$x^2 - 8x + \left(\frac{-8}{2}\right)^2 = x^2 - 8x + (-4)^2$

$= x^2 - 8x + 16$

$= (x - 4)^2$

b) $x^2 + 7x$

$\left(\frac{b}{2}\right)^2 = \left(\frac{7}{2}\right)^2$

$= \frac{49}{4}$

$x^2 + 7x + \left(\frac{7}{2}\right)^2 = x^2 + 7x + \frac{49}{4}$

$= \left(x + \frac{7}{2}\right)^2$

When completing the square, you need the leading coefficient *a* to be 1. If it is not 1, factor out *a* before completing the square.

Example 2: Solve $x^2 + 10x = 24$ by completing the square.

$x^2 + 10x = 24$

$\left(\frac{b}{2}\right)^2 = \left(\frac{10}{2}\right)^2$

$= 5^2$

$= 25$

$x^2 + 10x + 25 = 24 + 25$

$(x + 5)^2 = 49$

$x + 5 = \pm\sqrt{49}$

$x + 5 = \pm 7$

$x = -5 \pm 7$

$x = -5 + 7$ or $x = -5 - 7$

$x = 2$ or $x = -12$

Example 3: Solve $3x^2 - 6x - 9 = 0$ by completing the square.

$3x^2 - 6x = 9$

$3(x^2 - 2x) = 9$

$\left(\frac{b}{2}\right)^2 = \left(\frac{-2}{2}\right)^2$

$= (-1)^2$

$= 1$

$3(x^2 - 2x + 1) = 9 + 3$

$3(x - 1)^2 = 12$

$(x - 1)^2 = 4$

$x - 1 = \pm\sqrt{4}$

$x - 1 = \pm 2$

$x = 1 \pm 2$

$x = 1 + 2$ or $x = 1 - 2$

$x = 3$ or $x = -1$

Circles/Ellipses/Hyperbolas

A circle is a set of points in the xy-plane that are a fixed distance r from a fixed point (h, k). The fixed distance r is called the radius, and the fixed point (h, k) is called the center of the circle.

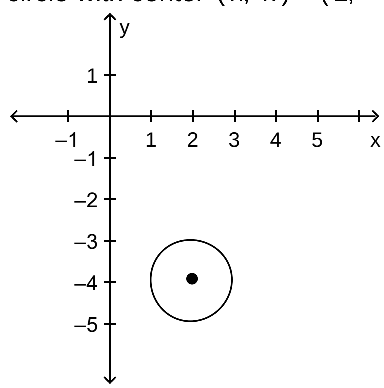
The standard form of an equation of a circle with radius r and center (h, k) is $(x-h)^2 + (y-k)^2 = r^2$.

When its graph is a circle, the equation $x^2 + y^2 + ax + by + c = 0$ is referred to as the general form of the equation of a circle. If an equation of a circle is in general form, use the method of completing the square to put the equation in standard form in order to identify its center and radius.

Example 4: Rewrite $x^2 + y^2 - 4x + 8y + 19 = 0$ in standard form and graph the equation.

$$x^2 + y^2 - 4x + 8y + 19 = 0$$
$$x^2 - 4x + y^2 + 8y = -19$$
$$x^2 - 4x + 4 + y^2 + 8y + 16 = -19 + 4 + 16$$
$$(x-2)^2 + (y+4)^2 = 1$$
$$(x-h)^2 + (y-k)^2 = r^2 \quad \Rightarrow h = 2, k = -4$$

This is a circle with center $(h, k) = (2, -4)$ and radius $r = 1$.



Example 5: Complete the square to find the center and the radius of the circle.
Given: $x^2 + y^2 - 6x + 10y + 18 = 0$

$$x^2 + y^2 - 6x + 10y + 18 = 0$$
$$x^2 - 6x + y^2 + 10y = -18$$
$$\left(\frac{b}{2}\right)^2 = \left(\frac{-6}{2}\right)^2 = (-3)^2 = 9 \qquad \left(\frac{b}{2}\right)^2 = \left(\frac{10}{2}\right)^2 = (5)^2 = 25$$
$$x^2 - 6x + 9 + y^2 + 10y + 25 = -18 + 9 + 25$$
$$(x-3)^2 + (y+5)^2 = 16$$
$$(x-3)^2 + (y+5)^2 = 4^2$$
$$(x-h)^2 + (y-k)^2 = r^2 \quad \Rightarrow h = 3, k = -5$$

This is a circle with center $(h, k) = (3, -5)$ and radius $r = 4$.

Ellipse Standard form, where $a > b$ and the center is (h, k):

Horizontal Ellipse

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Vertical Ellipse

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

Hyperbola Standard form, with center at (h, k):

Horizontal Transverse Axis

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Vertical Transverse Axis

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$