Section 6.3 – Polar-Cartesian Transformations – Day 2

Objective: Given a polar equation, transform it to Cartesian coordinates, and vice versa. If possible, identify the shape of the graph from its Cartesian equation.

For any polar curve, it is possible to find a Cartesian equation with the same curve for its graph. The polar equation can be transformed by substituting for r and θ in terms of x and y. Remember that $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$. So multiplying both equations by r, you get $x = r\cos\theta$ and $y = r\sin\theta$. By the Pythagorean Theorem, $x^2 + y^2 = r^2$, so, $r = \pm\sqrt{x^2 + y^2}$.

Polar-Cartesian Transformations: $x = r\cos\theta$ and $y = r\sin\theta$

$$r = \pm\sqrt{x^2 + y^2}$$

Example 2: Transform $r = 6\cos\theta$ to Cartesian coordinates. Identify the graph from its Cartesian equation.

$r = 6\cos\theta$

$r^2 = 6r\cos\theta$ Multiply both sides by r

$x^2 + y^2 = 6x$ Replace r^2 and $r\cos\theta$ with Cartesian coordinates

$x^2 - 6x + y^2 = 0$

$A = 1, C = 1$

$AC = 1(1)$

$= 1(1)$

$= 1$

> 0

$\Rightarrow$ an Ellipse and since $A = C$, it is a Circle

Example 3: Transform the ellipse $3x^2 + 4y^2 - 6x - 9 = 0$ to polar coordinates. Transform the resulting equation so that r is expressed explicitly in terms of θ.

$3x^2 + 4y^2 - 6x - 9 = 0$

$3(r\cos\theta)^2 + 4(r\sin\theta)^2 - 6r\cos\theta - 9 = 0$ Since $x = r\cos\theta$ and $y = r\sin\theta$

$3r^2\cos^2\theta + 4r^2\sin^2\theta - 6r\cos\theta - 9 = 0$

$(4-1)\, r^2\cos^2\theta + 4r^2\sin^2\theta - 6r\cos\theta - 9 = 0$

$4r^2\cos^2\theta + 4r^2\sin^2\theta - r^2\cos^2\theta - 6r\cos\theta - 9 = 0$ Make the $\sin^2\theta$ and $\cos^2\theta$ terms have equal coefficients

$4r^2\cos^2\theta + 4r^2\sin^2\theta - (r^2\cos^2\theta + 6r\cos\theta + 9) = 0$ Factor out -1

$4r^2(\cos^2\theta + \sin^2\theta) - (r\cos\theta + 3)^2 = 0$ By grouping and factoring

$4r^2(1) - (r\cos\theta + 3)^2 = 0$

$4r^2 - (r\cos\theta + 3)^2 = 0$

$4r^2 = (r\cos\theta + 3)^2$

$2r = r\cos\theta + 3$ Take the square root of each side

$2r - r\cos\theta = 3$

$r(2 - \cos\theta) = 3$

$r = \frac{3}{2 - \cos\theta}$

Section 6.3 – Polar-Cartesian Transformations – Day 2 (continued)

Example 4: Transform $r = \dfrac{3}{1+2\cos\theta}$ into Cartesian coordinates. If possible, identify the type of conic section.

$$r = \frac{3}{1+2\cos\theta}$$
$$r(1+2\cos\theta) = 3$$
$$r + 2r\cos\theta = 3$$
$$r + 2x = 3$$
$$r = 3 - 2x$$
$$r^2 = (3-2x)^2$$
$$x^2 + y^2 = (3-2x)^2$$
$$x^2 + y^2 = 9 - 12x + 4x^2$$
$$-3x^2 + y^2 + 12x - 9 = 0$$

$$A = -3,\ C = 1$$
$$AC = -3(1)$$
$$= -3$$
$$< 0$$
$\Rightarrow$ a Hyperbola

Example 5: Transform $r = \sin\theta\tan\theta$ into Cartesian coordinates. If possible, identify the type of conic section.

$$r = \sin\theta\tan\theta$$
$$r = \sin\theta\frac{\sin\theta}{\cos\theta}$$
$$r = \frac{\sin^2\theta}{\cos\theta}$$
$$r\cos\theta = \sin^2\theta$$
$$x = \left(\frac{y}{r}\right)^2$$
$$x = \frac{y^2}{r^2}$$
$$xr^2 = y^2$$
$$x(x^2 + y^2) = y^2$$
$x^3 + xy^2 - y^2 = 0$ Degree 3, so not a conic section

Example 6: Transform $y^2 = x$ into polar coordinates.

$$y^2 = x$$
$$(r\sin\theta)^2 = r\cos\theta$$
$$r^2\sin^2\theta = r\cos\theta$$
$$r = \frac{\cancel{r}\cos\theta}{\cancel{r}\sin^2\theta}$$
$$r = \frac{\cos\theta}{\sin^2\theta}$$
$$r = \frac{1}{\sin\theta}\left(\frac{\cos\theta}{\sin\theta}\right)$$
$$r = \csc\theta\cot\theta$$