Section 6.5 – Imaginary and Complex Numbers

The square root of a real number is nonnegative, so there is no real number x for which $x^2 = -1$. To remedy this, a number i whose square is -1 was introduced (by Euler). The number i is called the imaginary unit.

$$i = \sqrt{-1} \qquad i^2 = -1$$

$$i^3 = i^2(i) = (-1)\,i = -i \qquad i^5 = i^4(i) = (1)i = i \qquad i^7 = i^4(i^3) = (1)i^3 = -i$$

$$i^4 = i^2(i^2) = (-1)(-1) = 1 \qquad i^6 = i^4(i^2) = (1)(-1) = -1 \qquad i^8 = i^4(i^4) = (1)(1) = 1$$

$\Rightarrow$ The powers of i are cyclic and repeat in a pattern of four numbers.

So, $i^{76} = i^{4(19)}$
$= \left(i^4\right)^{19}$
$= (1)^{19}$
$= 1$

To simplify i^n, where n is an integer, divide n by 4. If the remainder is r, then $i^n = i^r$ where $r = 0, 1, 2,$ or 3.

Then use the pattern $i^0 = 1$, $i^1 = i$, $i^2 = -1$, and $i^3 = -i$.

So, $i^{66} = i^{4(16)+2}$
$= i^{4(16)} i^2$
$= \left(i^4\right)^{16} i^2$
$= (1)^{16}(-1)$
$= (1)(-1)$
$= -1$

OR

i^{66} : $66 \div 4 = 16$, $66 - 64 = 2$. So, $i^{66} \equiv i^2 = -1$

Complex Numbers

A complex number is any number that can be written in the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$. The real number a is called the real part of the number $a + bi$, and the real number b is called the imaginary part of the number $a + bi$.

The form $a + bi$ is called the standard form of a complex number. Real numbers are complex numbers for which $b = 0$. So, we write $a + 0i$ as a. A complex number is called a pure imaginary number if its real part, a, is 0. So, $0 + bi$, written as bi, is a pure imaginary number.

Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal.
$\Rightarrow$ $a + bi = c + di$ if and only if $a = c$ and $b = d$.

Operations with Complex Numbers

Adding, subtracting, and multiplying are done the same as for linear binomials, with $i^2 = -1$.

Addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$
Subtraction: $(a + bi) - (c + di) = (a - c) + (b - d)i$
Multiplication: $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$

Example 1: $(4 + 6i) + (-1 + 5i) = (4 + -1) + (6 + 5)i$
$= 3 + 11i$

$(3 + 7i) - (6 + 2i) = (3 - 6) + (7 - 2)i$
$= -3 + 5i$

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Example 2: $(2+3i)(4+5i) = 2(4)+2(5i)+3i(4)+3i(5i)$

$$= 8+10i+12i+15i^2$$
$$= 8+22i+15(-1)$$
$$= 8+22i-15$$
$$= -7+22i$$

The conjugate of a complex number $a+bi$ is $a-bi$. The conjugate of $a+bi$ is denoted by $\overline{a+bi}$.

Example: $\overline{2+5i} = 2-5i$ $\quad\overline{-1-3i} = -1+3i$

Theorem: The product of a complex number and its conjugate is a nonnegative real number.
Thus, if $z = a+bi$, then $z\bar{z} = a^2+b^2$.

Example 3: $z = 2+3i$, $\bar{z} = 2-3i$

$$z\bar{z} = (2+3i)(2-3i) \qquad \text{OR} \qquad z\bar{z} = 2^2+3^2$$
$$= 4-6i+6i-9i^2 \qquad\qquad = 4+9$$
$$= 4-9(-1) \qquad\qquad = 13$$
$$= 13$$

To simplify a quotient with an imaginary number in the denominator, multiply by a fraction equal to 1, using the conjugate of the denominator. This process is called rationalizing the denominator.

Example 4: Simplify $\frac{3+5i}{2-3i}$.

$$\frac{3+5i}{2-3i} = \frac{3+5i}{2-3i}\left(\frac{2+3i}{2+3i}\right)$$
$$= \frac{6+9i+10i+15i^2}{4+6i-6i-9i^2}$$
$$= \frac{6+19i+15(-1)}{4-9(-1)}$$
$$= \frac{-9+19i}{13}$$
$$= \frac{-9}{13}+\frac{19}{13}i$$

Never leave an imaginary number in a denominator – always rationalize the denominator.

Square Roots of Negative Numbers

If N is a positive real number, we define the principal square root of $-N$ as $\sqrt{-N} = \sqrt{N}\,i$.

Example 5: $\sqrt{-3} = \sqrt{3}\,i$ $\quad\sqrt{-12} = \sqrt{12}\,i$

$$= \sqrt{4}\sqrt{3}\,i$$
$$= 2\sqrt{3}\,i$$

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Example 6: Solve the equation $x^2 + 16 = 0$.

$$x^2 + 16 = 0$$
$$x^2 = -16$$
$$x = \pm\sqrt{-16}$$
$$= \pm\sqrt{16}\, i$$
$$= \pm 4i$$

Quadratic equations with a negative discriminant have no real solution. But if we extend our number system to allow complex numbers, quadratic equations will always have a solution.

Theorem – In the complex number system, the solutions of the quadratic equation $ax^2 + bx + c = 0$,

where a, b, and c are real numbers and $a \neq 0$, are given by the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Example 7: Solve the equation $x^2 - 6x + 10 = 0$ in the complex number system.

$\Rightarrow$ $a = 1$, $b = -6$, $c = 10$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(10)}}{2(1)}$$
$$= \frac{6 \pm \sqrt{36 - 40}}{2}$$
$$= \frac{6 \pm \sqrt{-4}}{2}$$
$$= \frac{6 \pm \sqrt{4}\, i}{2}$$
$$= \frac{6 \pm 2i}{2}$$
$$= 3 \pm i$$
$$x = \{\, 3 + i,\ 3 - i \,\}$$

The discriminant gives you information about the types of solutions. In the complex number system, a quadratic equation $ax^2 + bx + c = 0$ with real coefficients has the following types of solutions:

1) If $b^2 - 4ac > 0$, the equation has two unequal real solutions.
2) If $b^2 - 4ac = 0$, the equation has one repeated real solution, or a double root.
3) If $b^2 - 4ac < 0$, the equation has two complex solutions that are not real.
   The solutions are conjugates of each other.

For Example 7 above, $b^2 - 4ac = (-6)^2 - 4(1)(10)$.

$$= 36 - 40$$
$$= -4$$
$$< 0$$

So you know that the solutions are complex numbers, and they are complex conjugates of each other.