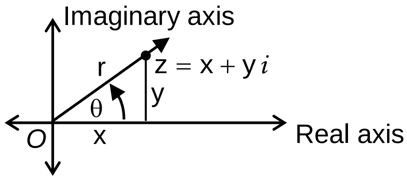
Section 6.6 – Complex Numbers in Polar Form – Day 1

Objective:

1) Given a complex number $x + yi$, transform it to polar form, and vice versa.

Recall we can graph a complex number $z = x + yi$ in the complex plane.



When a complex number is written in the standard form $z = x + yi$ we say that it is in rectangular, or Cartesian, form.
If (r, θ) are the polar coordinates of this point, then $x = r\cos\theta$ and $y = r\sin\theta$.

For a complex number $z = x + yi$, its polar form is

$$\begin{aligned} z &= x + yi \\ &= r\cos\theta + r\sin\theta\, i \\ &= r(\cos\theta + i\sin\theta) \\ &= r\,\text{cis}\theta, \quad \text{(abbreviated form)} \end{aligned}$$

where $r = \sqrt{x^2 + y^2}$ is the modulus of z, and θ is the smallest nonnegative angle in standard position to the radius vector.

If $z = r(\cos\theta + i\sin\theta)$ is the polar form of a complex number, the angle θ, $0^\circ \le \theta < 360^\circ$, is called the argument of z.

Let $z = x + yi$ be a complex number. The magnitude or modulus of z, denoted by $|z|$ or r, is defined as the distance from the origin to the point (x, y). Thus, $|z| = \sqrt{x^2 + y^2}$ or $r = \sqrt{x^2 + y^2}$, since

$$\begin{aligned} |z| &= \sqrt{x^2 + y^2} \\ &= \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} \\ &= \sqrt{r^2(\cos^2\theta + \sin^2\theta)} \\ &= \sqrt{r^2(1)} \\ &= \sqrt{r^2} \\ &= r \end{aligned}$$

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You need to be able to convert from rectangular coordinates to polar coordinates, and vice versa. Remember that the origin in rectangular coordinates is the pole in polar coordinates and that the positive x-axis in rectangular coordinates is the polar axis in polar coordinates.

Theorem: Conversion from polar coordinates to rectangular coordinates.
If P is a point with polar coordinates (r, θ), the rectangular coordinates (x, y) of P are given by $x = r\cos\theta$ and $y = r\sin\theta$.

Example 1: Find the rectangular coordinates of the points with the following polar coordinates:

a) $(r, \theta) = \left(4, \frac{\pi}{3}\right)$ b) $(r, \theta) = \left(-2, \frac{-\pi}{6}\right)$

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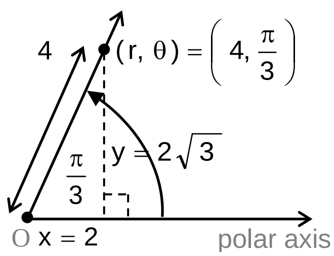


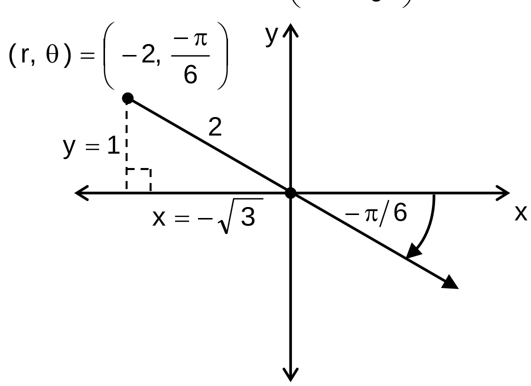
$$\begin{aligned} x &= r\cos\theta \\ &= 4\cos\left(\frac{\pi}{3}\right) \\ &= 4\left(\frac{x}{r}\right) \\ &= 4\left(\frac{1/2}{1}\right) \\ &= 4\left(\frac{1}{2}\right) \\ &= 2 \end{aligned}$$

$$\begin{aligned} y &= r\sin\theta \\ &= 4\sin\left(\frac{\pi}{3}\right) \\ &= 4\left(\frac{y}{r}\right) \\ &= 4\left(\frac{\sqrt{3}/2}{1}\right) \\ &= 4\left(\frac{\sqrt{3}}{2}\right) \\ &= 2\sqrt{3} \end{aligned}$$

Now, $\frac{\pi}{3} : (x, y) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

Thus, polar: $(r, \theta) = \left(4, \frac{\pi}{3}\right)$ rectangular: $(x, y) = \left(2, 2\sqrt{3}\right)$

b) $(r, \theta) = \left(-2, \frac{-\pi}{6}\right)$



$$\begin{aligned} x &= r\cos\theta \\ &= -2\cos\left(\frac{-\pi}{6}\right) \\ &= -2\cos\left(\frac{\pi}{6}\right), \text{ cos even} \\ &= -2\left(\frac{x}{r}\right) \quad \text{Now, } \frac{\pi}{6} : \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \\ &= -2\left(\frac{\sqrt{3}/2}{1}\right) \\ &= -2\left(\frac{\sqrt{3}}{2}\right) \\ &= -\sqrt{3} \end{aligned}$$

$$\begin{aligned} y &= r\sin\theta \\ &= -2\sin\left(\frac{-\pi}{6}\right) \\ &= -2\left[-\sin\left(\frac{\pi}{6}\right)\right], \text{ sin odd} \\ &= 2\sin\left(\frac{\pi}{6}\right) \\ &= 2\left(\frac{y}{r}\right) \\ &= 2\left(\frac{1/2}{1}\right) \\ &= 2\left(\frac{1}{2}\right) \\ &= 1 \end{aligned}$$

Thus, polar: $(r, \theta) = \left(-2, \frac{-\pi}{6}\right)$ rectangular: $(x, y) = \left(-\sqrt{3}, 1\right)$

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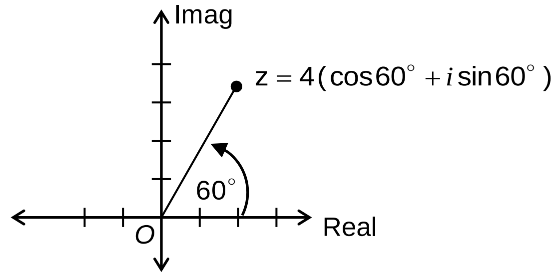
Example 2: Plot the point corresponding to $z = 4(\cos 60^\circ + i \sin 60^\circ)$ in the complex plane, and write an expression for z in rectangular form.

$$z = 4(\cos 60^\circ + i\sin 60^\circ)$$
$$= r(\cos\theta + i\sin\theta)$$
$$\Rightarrow (r, \theta) = (4, 60^\circ)$$

But for 60° : $(x, y) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, so in rectangular form

$$z = 4(\cos 60^\circ + i\sin 60^\circ)$$
$$= 4\left(\frac{x}{r} + i\,\frac{y}{r}\right)$$
$$= 4\left(\frac{1/2}{1} + i\,\frac{\sqrt{3}/2}{1}\right)$$
$$= 4\left(\frac{1}{2} + i\,\frac{\sqrt{3}}{2}\right)$$
$$= 2 + 2\sqrt{3}\,i$$

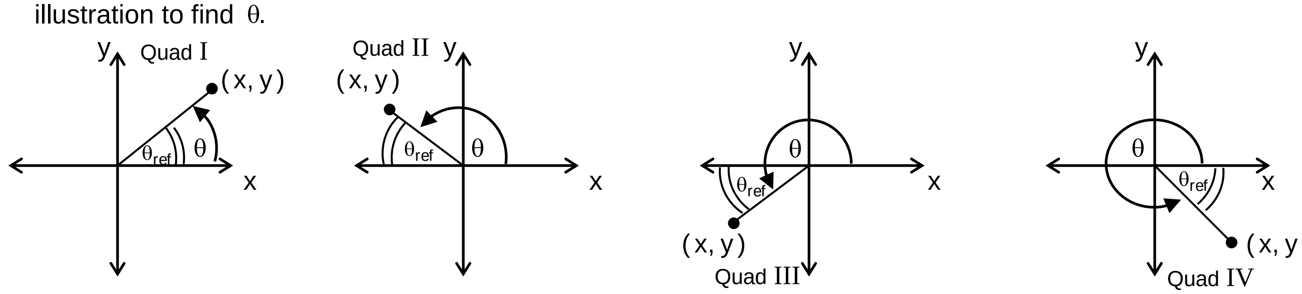


Rectangular coordinates of z: $(x, y) = (2, 2\sqrt{3})$
$\approx (2, 3.46)$

Converting from rectangular coordinates (x, y) to polar coordinates (r, θ) is a little more complicated than converting from polar to rectangular coordinates.

Steps for converting from rectangular to polar coordinates:

1) Plot the point (x, y).
2) To find r, compute the distance from the origin to point (x, y).
3) To find positive θ, it is best to compute the reference angle θ_{ref} of θ, then $\tan\theta_{ref} = \left|\frac{y}{x}\right|$, if $x \neq 0$, and use your illustration to find θ.



Quad I	Quad II	Quad III	Quad IV
$r = \sqrt{x^2 + y^2}$	$r = \sqrt{x^2 + y^2}$	$r = \sqrt{x^2 + y^2}$	$r = \sqrt{x^2 + y^2}$
$\theta = \theta_{ref}$	$\theta = \pi - \theta_{ref}$	$\theta = \pi + \theta_{ref}$	$\theta = 2\pi - \theta_{ref}$
$= \tan^{-1}\left\lvert\frac{y}{x}\right\rvert$	$= \pi - \tan^{-1}\left\lvert\frac{y}{x}\right\rvert$	$= \pi + \tan^{-1}\left\lvert\frac{y}{x}\right\rvert$	$= 2\pi - \tan^{-1}\left\lvert\frac{y}{x}\right\rvert$

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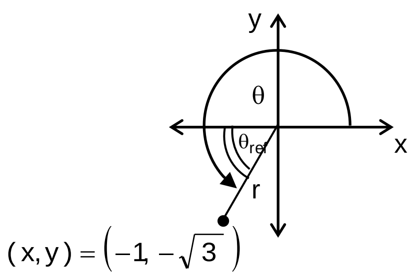
Example 3: Find the polar coordinates of a point whose rectangular coordinates are $(x, y) = \left(-1, -\sqrt{3}\right)$.



The distance r from the origin to the point $(x, y) = \left(-1, -\sqrt{3}\right)$ is

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{(-1)^2 + \left(-\sqrt{3}\right)^2} \\ &= \sqrt{1+3} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

Find the reference angle θ_{ref} :

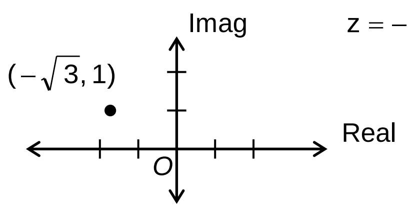
$$\begin{aligned} \theta_{ref} &= \tan^{-1}\left|\frac{y}{x}\right| \\ &= \tan^{-1}\left|\frac{-\sqrt{3}}{-1}\right| \\ &= \tan^{-1}\left(\sqrt{3}\right) \\ &= \frac{\pi}{3} \end{aligned}$$

$(x, y) = \left(-1, -\sqrt{3}\right)$ is in Quadrant III, thus,

$$\begin{aligned} \theta &= \pi + \theta_{ref} \\ &= \pi + \frac{\pi}{3} \\ &= \frac{4\pi}{3} \end{aligned}$$

Polar coordinates: $(r, \theta) = \left(2, \frac{4\pi}{3}\right)$ $\left(\text{or } \left(-2, \frac{\pi}{3}\right) \text{ or } \left(2, \frac{-2\pi}{3}\right) \text{ or } \ldots \right.$ An infinite number of possibilities)

Example 4: Plot the point corresponding to $z = -\sqrt{3} + i$ in the complex plane, and write an expression for z in polar form.



$z = -\sqrt{3} + i$ has rectangular coordinates $(-\sqrt{3}, 1) \approx (-1.73, 1)$.

$x = -\sqrt{3}$, $y = 1 \Rightarrow$ z is in Quadrant II

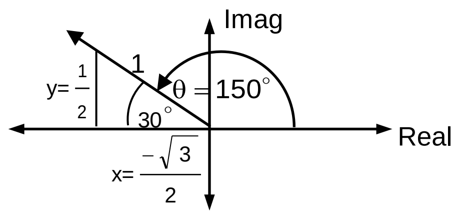
$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{\left(-\sqrt{3}\right)^2 + (1)^2} \\ &= \sqrt{3+1} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

Find the reference angle θ_{ref} :

$$\begin{aligned} \theta_{ref} &= \tan^{-1}\left|\frac{y}{x}\right| \\ &= \tan^{-1}\left|\frac{1}{-\sqrt{3}}\right| \\ &= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \\ &= \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) \\ &= 30^\circ \end{aligned}$$

$(x, y) = \left(-\sqrt{3}, 1\right)$ is in Quadrant II, thus,

$$\begin{aligned} \theta &= 180^\circ - \theta_{ref} \\ &= 180^\circ - 30^\circ \\ &= 150^\circ \end{aligned}$$



So, the polar form of $z = -\sqrt{3} + i$ is

$$\begin{aligned} z &= r(\cos\theta + i\sin\theta) \\ &= 2(\cos 150^\circ + i\sin 150^\circ) \\ &= 2\,\text{cis}\,150^\circ \end{aligned}$$