Section 6.6 – Complex Numbers in Polar Form – Day 2

Objective:

2) Given complex numbers in polar form, find products, quotients, and powers.

The polar form of a complex number provides an alternative for finding products and quotients of complex numbers.

Theorem:

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex numbers.

Then $z_1 z_2 = r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$

$= r_1 r_2 \operatorname{cis}(\theta_1+\theta_2)$

If $z_2 \neq 0$, then $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$

$= \dfrac{r_1}{r_2}\operatorname{cis}(\theta_1-\theta_2)$

Example 5: If $z = 2(\cos 40^\circ + i\sin 40^\circ)$ and $w = 6(\cos 20^\circ + i\sin 20^\circ)$, find zw and $\dfrac{z}{w}$. Leave your answers in polar form.

$$zw = [2(\cos 40^\circ + i\sin 40^\circ)]\,[6(\cos 20^\circ + i\sin 20^\circ)]$$
$$= 2(6)[\cos(40^\circ+20^\circ) + i\sin(40^\circ+20^\circ)]$$
$$= 12(\cos 60^\circ + i\sin 60^\circ)$$

$$\frac{z}{w} = \frac{2(\cos 40^\circ + i\sin 40^\circ)}{6(\cos 20^\circ + i\sin 20^\circ)}$$
$$= \frac{2}{6}[\cos(40^\circ-20^\circ) + i\sin(40^\circ-20^\circ)]$$
$$= \frac{1}{3}(\cos 20^\circ + i\sin 20^\circ)$$

Example 6: If $z = 2(\cos 340^\circ + i\sin 340^\circ)$ and $w = 6(\cos 50^\circ + i\sin 50^\circ)$, find zw and $\dfrac{z}{w}$. Leave your answers in polar form.

$$zw = [2(\cos 340^\circ + i\sin 340^\circ)]\,[6(\cos 50^\circ + i\sin 50^\circ)]$$
$$= 2(6)[\cos(340^\circ+50^\circ) + i\sin(340^\circ+50^\circ)]$$

$= 12(\cos 390^\circ + i\sin 390^\circ)$ Not polar form, since θ is not between 0° and 360°

$$= 12[\cos(360^\circ+30^\circ) + i\sin(360^\circ+30^\circ)]$$

$= 12(\cos 30^\circ + i\sin 30^\circ)$ by the periodic property

$$\frac{z}{w} = \frac{2(\cos 340^\circ + i\sin 340^\circ)}{6(\cos 50^\circ + i\sin 50^\circ)}$$
$$= \frac{2}{6}[\cos(340^\circ-50^\circ) + i\sin(340^\circ-50^\circ)]$$
$$= \frac{1}{3}(\cos 290^\circ + i\sin 290^\circ)$$

Section 6.6 – Complex Numbers in Polar Form – Day 2 (continued)

De Moivre's Theorem is a formula for raising a complex number z to the power n, where $n \geq 1$ is a positive integer.

Theorem – De Moivre's Theorem

If $z = r(\cos\theta + i\sin\theta)$ is a complex number, then $z^n = r^n\left(\cos(n\theta) + i\sin(n\theta)\right)$ where $n \geq 1$ is a positive integer.

Example 7: Write $\left[2(\cos 40^\circ + i\sin 40^\circ)\right]^3$ in the standard form $a + bi$.

$$\left[2(\cos 40^\circ + i\sin 40^\circ)\right]^3 = 2^3\left[\cos\left(3(40^\circ)\right) + i\sin\left(3(40^\circ)\right)\right] \quad \text{by De Moivre's Theorem}$$
$$= 8(\cos 120^\circ + i\sin 120^\circ)$$
$$= 8\left(\frac{x}{r} + \frac{y}{r}i\right) \qquad \text{Now, } 120^\circ:\ (x, y) = \left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right).$$
$$= 8\left(\frac{-1/2}{1} + \frac{\sqrt{3}/2}{1}i\right)$$
$$= 8\left(\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right)$$
$$= -4 + 4\sqrt{3}\, i$$

Example 8: Use De Moivre's Theorem to write $(-4 + 3i)^3$ in the standard form $a + bi$.

$-4 + 3i$: rectangular coordinates $(x,\ y) = (-4,\ 3)$ $\Rightarrow$ Quadrant II

$$r = \sqrt{x^2 + y^2}$$
$$= \sqrt{(-4)^2 + (3)^2}$$
$$= \sqrt{16 + 9}$$
$$= \sqrt{25}$$
$$= 5$$

Find the reference angle θ_{ref} : $\theta_{ref} = \tan^{-1}\left|\frac{y}{x}\right|$ $\qquad$ θ is in Quadrant II, thus, $\theta = 180^\circ - \theta_{ref}$

$$= \tan^{-1}\left|\frac{3}{-4}\right| \qquad \theta = 180^\circ - 36.9^\circ$$
$$= \tan^{-1}\left(\frac{3}{4}\right) \qquad \theta \approx 143.1^\circ$$
$$= 36.86^\circ$$
$$\approx 36.9^\circ$$

Thus, the polar form of $-4 + 3i$ is $-4 + 3i = r(\cos\theta + i\sin\theta)$

$$= 5(\cos 143.1^\circ + i\sin 143.1^\circ)$$

So, $(-4 + 3i)^3 = \left[5(\cos 143.1^\circ + i\sin 143.1^\circ)\right]^3$

$$= 5^3\left[\cos\left(3(143.1^\circ)\right) + i\sin\left(3(143.1^\circ)\right)\right] \quad \text{by De Moivre's Theorem}$$
$$= 125(\cos 429.3^\circ + i\sin 429.3^\circ)$$
$$= 125\left[\cos\left(69.3^\circ + 360^\circ\right) + i\sin\left(69.3^\circ + 360^\circ\right)\right]$$
$$= 125(\cos 69.3^\circ + i\sin 69.3^\circ) \quad \text{by the periodic property}$$
$$\approx 125(0.353475 + 0.935444\, i)$$
$$= 44.184375 + 116.9305\, i$$
$$\approx 44 + 117i$$