Section 6.6 – Complex Numbers in Polar Form – Day 3

Objective:

3) Given complex numbers in polar form, find roots.

Complex Roots

Let w be a given complex number, and let $n \geq 2$ denote a positive integer. Any complex number z that satisfies the equation $z^n = w$ is called a complex n^{th} root of w. So, solutions of $z^2 = w$ are called the complex square roots of w and solutions of $z^3 = w$ are called the complex cube roots of w.

Theorem – Finding Complex Roots

Let $w = r(\cos\theta + i\sin\theta)$ be a complex number. If $w \neq 0$, there are n distinct complex n^{th} roots of w, given by the formula $z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{360^\circ k}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{360^\circ k}{n}\right)\right]$, where $k = 0, 1, 2, \ldots, n-1$.

Example 9: Find the three complex cube roots of $1-\sqrt{3}\,i$.

$1-\sqrt{3}\,i$: rectangular coordinates $(x, y) = (1, -\sqrt{3})$ $\Rightarrow$ Quadrant IV

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{(1)^2 + (-\sqrt{3})^2} \\ &= \sqrt{1+3} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

Find the reference angle θ_{ref} : $\theta_{ref} = \tan^{-1}\left|\frac{y}{x}\right|$

$$\begin{aligned} &= \tan^{-1}\left|\frac{-\sqrt{3}}{1}\right| \\ &= \tan^{-1}\left|-\sqrt{3}\right| \\ &= \tan^{-1}\left(\sqrt{3}\right) \\ &= 60^\circ \end{aligned}$$

θ is in Quadrant IV, thus, $\theta = 360^\circ - \theta_{ref}$

$$\begin{aligned} \theta &= 360^\circ - 60^\circ \\ \theta &= 300^\circ \end{aligned}$$

Thus, the polar form of $1-\sqrt{3}\,i$ is

$$\begin{aligned} 1-\sqrt{3}\,i &= r(\cos\theta + i\sin\theta) \\ &= 2(\cos 300^\circ + i\sin 300^\circ) \end{aligned}$$

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From the complex root theorem, the three complex cube roots of $1-\sqrt{3}\,i$ are

$$z_k = \sqrt[3]{2}\left[\cos\left(\frac{300^\circ}{3}+\frac{360^\circ k}{3}\right)+i\sin\left(\frac{300^\circ}{3}+\frac{360^\circ k}{3}\right)\right],\quad k=0,\ 1,\ 2$$
$$= \sqrt[3]{2}\left[\cos\left(100^\circ+120^\circ k\right)+i\sin\left(100^\circ+120^\circ k\right)\right],\quad k=0,\ 1,\ 2$$

So,

$$z_0 = \sqrt[3]{2}\left[\cos(100^\circ)+i\sin(100^\circ)\right]$$
$$z_1 = \sqrt[3]{2}\left[\cos(100^\circ+120^\circ)+i\sin(100^\circ+120^\circ)\right]$$
$$z_2 = \sqrt[3]{2}\left[\cos(100^\circ+2(120^\circ))+i\sin(100^\circ+2(120^\circ))\right]$$
$$= \sqrt[3]{2}\left[\cos(100^\circ+240^\circ)+i\sin(100^\circ+240^\circ)\right]$$

Thus the three complex cube roots of $1-\sqrt{3}\,i$ are

$$z_0 = \sqrt[3]{2}\left[\cos(100^\circ)+i\sin(100^\circ)\right]$$
$$z_1 = \sqrt[3]{2}\left[\cos(220^\circ)+i\sin(220^\circ)\right]$$
$$z_2 = \sqrt[3]{2}\left[\cos(340^\circ)+i\sin(340^\circ)\right]$$