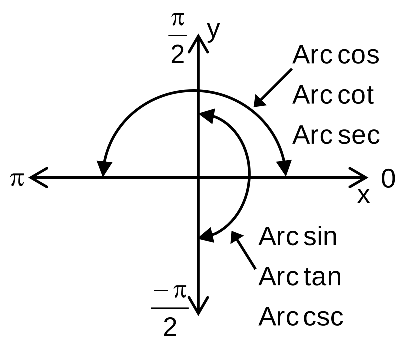
Section 4.6 – Exact Values of Inverse Functions

Objective: Be able to find exact values of special inverse circular functions.

In sections 1.4 and 2.5 you found exact values of functions for special arguments. Here you will do the same sort of thing for inverse circular functions.

Remember, $\cos^{-1} x$ is the same as $\arccos x$.

The most important thing for you to know is the range of each inverse function. It often helps to think of the inverse function as an arc or angle in an xy-coordinate system.



If the argument of the inverse function is positive, for all six trig functions such as $\operatorname{Arc}\sin 0.7$, then the arc or angle is in Quadrant I. If the argument of the inverse function is negative, the arc or angle is in Quadrant II for Arccos, Arccot, and Arcsec, and it is in Quadrant IV for Arcsin, Arctan, and Arccsc. None of the inverse functions correspond to an arc or angle in Quadrant III.

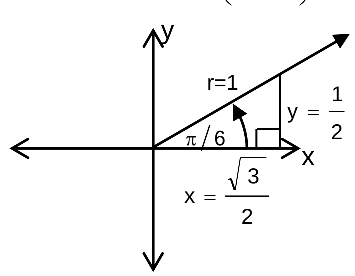
Recall: Ranges of Inverse Circular Functions:

$$y = \operatorname{Arc}\sin x \equiv \operatorname{Sin}^{-1}x, \quad y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$$

$$y = \operatorname{Arc}\cos x \equiv \operatorname{Cos}^{-1}x, \quad y \in [0, \pi]$$

$$y = \operatorname{Arc}\tan x \equiv \operatorname{Tan}^{-1}x, \quad y \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$$

Example 1: Evaluate $\operatorname{Cos}^{-1}\left(\frac{\sqrt{3}}{2}\right)$.



Let $\theta = \operatorname{Cos}^{-1}\left(\frac{\sqrt{3}}{2}\right)$, where $\theta \in [0, \pi]$.

Since the argument is positive, angle θ must be in Quadrant I.

From the graph, you can see that

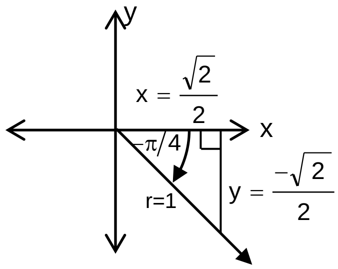
$$\cos\frac{\pi}{6} = \frac{x}{r} = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2}$$

Thus,

$$\theta = \operatorname{Cos}^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

Section 4.6 – Exact Values of Inverse Functions (continued)

Example 2: Evaluate $\text{Arc}\sin\left(\dfrac{-\sqrt{2}}{2}\right)$.



Let $\theta = \text{Arc}\sin\left(\dfrac{-\sqrt{2}}{2}\right)$, where $\theta \in \left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$.

Since the argument is negative, angle θ must be in Quadrant IV.

From the graph, you can see that $\sin\dfrac{-\pi}{4} = \dfrac{y}{r}$

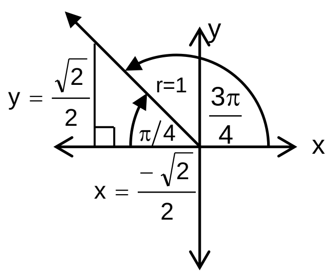
$$= \frac{\frac{-\sqrt{2}}{2}}{1}$$

$$= \frac{-\sqrt{2}}{2}$$

Thus, $\theta = \text{Arc}\sin\left(\dfrac{-\sqrt{2}}{2}\right)$

$$= \frac{-\pi}{4}$$

Example 3: Evaluate $\text{Arc}\cos\left(\dfrac{-\sqrt{2}}{2}\right)$.



Let $\theta = \text{Arc}\cos\left(\dfrac{-\sqrt{2}}{2}\right)$, where $\theta \in [0, \pi]$.

Since the argument is negative, angle θ must be in Quadrant II.

From the graph, you can see that $\cos\dfrac{3\pi}{4} = \dfrac{x}{r}$

$$= \frac{\frac{-\sqrt{2}}{2}}{1}$$

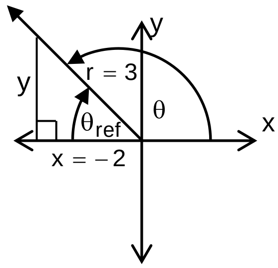
$$= \frac{-\sqrt{2}}{2}$$

Thus, $\theta = \text{Arc}\cos\left(\dfrac{-\sqrt{2}}{2}\right)$

$$= \frac{3\pi}{4}$$

Section 4.6 – Exact Values of Inverse Functions (continued)

Example 4: Evaluate $\tan\left(\text{Arc}\cos\left(\frac{-2}{3}\right)\right)$.



Let $\theta = \text{Arc}\cos\left(\frac{-2}{3}\right)$, where $\theta \in [0, \pi]$.

Since the argument is negative, angle θ must be in Quadrant II.

Now, $\cos\theta = \frac{-2}{3}$

$= \frac{x}{r} \Rightarrow$ Assume $x = -2$, $r = 3$. Find y.

Know $x^2 + y^2 = r^2$

$(-2)^2 + y^2 = 3^2$

$4 + y^2 = 9$

$y^2 = 5$

$y = \pm\sqrt{5}$

θ in Quadrant II, so y is positive and $y = \sqrt{5}$

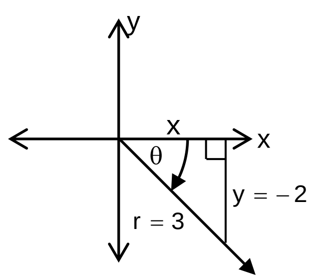
So, $\tan\left(\text{Arc}\cos\left(\frac{-2}{3}\right)\right) = \tan\theta$

$= \frac{y}{x}$

$= \frac{\sqrt{5}}{-2}$

$= \frac{-\sqrt{5}}{2}$

Example 5: Evaluate $\tan\left(\text{Arc}\sin\left(\frac{-2}{3}\right)\right)$.



Let $\theta = \text{Arc}\sin\left(\frac{-2}{3}\right)$, where $\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

Since the argument is negative, angle θ must be in Quadrant IV.

Now, $\sin\theta = \frac{-2}{3}$

$= \frac{y}{r} \Rightarrow$ Assume $y = -2$, $r = 3$. Find x.

Know $x^2 + y^2 = r^2$

$x^2 + (-2)^2 = 3^2$

$x^2 + 4 = 9$

$x^2 = 5$

$x = \pm\sqrt{5}$

θ in Quadrant IV, so x is positive and $x = \sqrt{5}$

So, $\tan\left(\text{Arc}\sin\left(\frac{-2}{3}\right)\right) = \tan\theta$

$= \frac{y}{x}$

$= \frac{-2}{\sqrt{5}}$

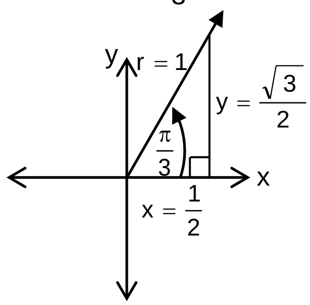
$= \frac{-2}{\sqrt{5}}\left(\frac{\sqrt{5}}{\sqrt{5}}\right)$

$= \frac{-2\sqrt{5}}{5}$

Section 4.6 – Exact Values of Inverse Functions (continued)

Example 6: Evaluate $\text{Cos}^{-1}\left(\cos\frac{\pi}{3}\right)$.

First, find $\cos\frac{\pi}{3}$.



$$\cos\frac{\pi}{3} = \frac{x}{r} = \frac{\frac{1}{2}}{1} = \frac{1}{2}$$

So, $\text{Cos}^{-1}\left(\cos\frac{\pi}{3}\right) = \text{Cos}^{-1}\left(\frac{1}{2}\right)$

Let $\theta = \text{Cos}^{-1}\left(\frac{1}{2}\right)$, where $\theta \in [0, \pi]$.

Since the argument is positive, angle θ must be in Quadrant I.

From the graph, you can see that $\cos\frac{\pi}{3} = \frac{x}{r} = \frac{\frac{1}{2}}{1} = \frac{1}{2}$

Thus, $\theta = \text{Cos}^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

So, $\text{Cos}^{-1}\left(\cos\frac{\pi}{3}\right) = \text{Cos}^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

*** In Example 6, $\text{Cos}^{-1}(\cos x)$ turned out to be x. This is not always true. This only happens if x is in the range of the inverse cosine function.

Note that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$ provided x is in the range of the outside function, and in the domain of the inside function.

For example, $\text{Cos}^{-1}(\cos(2\pi)) = \text{Cos}^{-1}\left(\frac{x}{r}\right)$ $\quad 2\pi : (x, y) = (1, 0)$

$= \text{Cos}^{-1}\left(\frac{1}{1}\right)$

$= \text{Cos}^{-1}(1)$ $\quad 0 : (x, y) = (1, 0)$

$= 0$ Not 2π, since it must be an angle θ such that $0 \le \theta \le \pi$

All material has been taken from *Trigonometry,* by P. Foerster, 3rd Edition