Appendix A.9 – Interval Notation; Solving Inequalities

Inequalities

The symbols $<$, $>$, $\leq$, and $\geq$ are called **inequality symbols**. An **inequality** is a statement in which two expressions are related by an inequality symbol. The expressions are referred to as the sides of the inequality. Statements of the form $a < b$ or $b > a$ are called **strict inequalities**. Statements of the form $a \leq b$ or $b \geq a$ are called **nonstrict inequalities**. Note that $a < b$ and $b > a$ mean the same thing.

$a > 0$ is equivalent to a is positive.
$a < 0$ is equivalent to a is negative.

The inequality $a > 0$ is sometimes read as "a is positive." If $a \geq 0$, then either $a > 0$ or $a = 0$, and this is read as "a is nonnegative."

Suppose that *a* and *b* are two real numbers and $a < b$. The notation $a < x < b$ means that x is a number between *a* and *b*. The expression $a < x < b$ is equivalent to the two inequalities $a < x$ and $x < b$. Similarly, the expression $a \leq x \leq b$ is equaivalent to the two inequalitites $a \leq x$ and $x \leq b$. The remaining two possibilities, $a \leq x < b$ and $a < x \leq b$, are defined similarly.

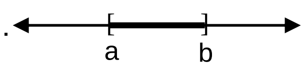
Although it is acceptable to write $4 \geq x \geq 3$, it is preferable to reverse the inequality symbols and write instead $3 \leq x \leq 4$ so that the values go from smaller to larger, reading from left to right.

A statement such as $2 \leq x \leq 1$ is false because there is no number x for which $2 \leq x$ and $x \leq 1$. Finally, never mix inequality symbols, as in $2 \leq x \geq 3$.

Use Interval Notation

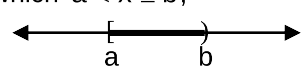
Let a and b represent two real numbers with $a < b$.

A **closed interval**, denoted by $[a, b]$, consists of all real numbers x for which $a \leq x \leq b$.



An **open interval**, denoted by (a, b), consists of all real numbers x for which $a < x < b$.

The **half-open**, or **half-closed, intervals** are $(a, b]$, consisting of all real numbers x for which $a < x \leq b$, and $[a, b)$, consisting of all real numbers x for which $a \leq x < b$.



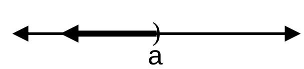
In each of these definitions, *a* is called the **left endpoint** and *b* the **right endpoint** of the inverval.

The symbol ∞ (infinity) is not a real number, but a notation used to indicate unboundedness in the positive direction. The symbol $-\infty$ (negative infinity) also is not a real number, but a notation used to indicate unboundedness in the negative direction.

Using ∞ and $-\infty$, we can define five other kinds of intervals:

- $[a, \infty)$ Consists of all real numbers x for which $x \geq a$ $(a \leq x < \infty)$

- (a, ∞) Consists of all real numbers x for which $x > a$ $(a < x < \infty)$
- $(-\infty, a]$ Consists of all real numbers x for which $x \leq a$ $(-\infty < x \leq a)$
- $(-\infty, a)$ Consists of all real numbers x for which $x < a$ $(-\infty < x < a)$



- $(-\infty, \infty)$ Consists of all real numbers x $(-\infty < x < \infty)$

Note that $-\infty$ and ∞ are never included as endpoints, since neither is a real number.

Appendix A.9 – Interval Notation; Solving Inequalities (continued)

Example 1: Write each inequality using interval notation.

a) $4 \le x \le 6$ b) $-6 < x < 2$ c) $x > 9$ d) $x \le 5$

a) $4 \le x \le 6$ describes all real numbers x between 4 and 6, inclusive. In interval notation, $4 \le x \le 6$ is written $[4,6]$.

b) In interval notation, $-6 < x < 2$ is written $(-6,2)$.

c) In interval notation, $x > 9$ is written $(9,\infty)$.

d) In interval notation, $x \le 5$ is written $(-\infty,5]$.

Example 2: Write each interval as an inequality involving x.

a) $[2,7)$ b) $(6,\infty)$ c) $[-3,5]$ d) $(-\infty,4]$

a) $[2,7)$ consists of all real numbers x for which $2 \le x < 7$.

b) $(6,\infty)$ consists of all real numbers x for which $x > 6$.

c) $[-3,5]$ consists of all real numbers x for which $-3 \le x \le 5$.

d) $(-\infty,4]$ consists of all real numbers x for which $x \le 4$.

Use Properties of Inequalities

The product of two positive real numbers is positive, the product of two negative real numbers is positive, and the product of 0 and 0 is 0. For any real number a, the value of a^2 is 0 or positive; that is, a^2 is nonnegative. This is called the **nonnegative property.**

Nonnegative Property: For any real number *a*, $a \ge 0$.

When the same number is added to both sides of an inequality, an equivalent inequality is obtained. For example, since $4 < 7$, then $4+3 < 7+3$ or $7 < 10$. This is called the addition property of inequalities.

Addition Property of Inequalities: For real numbers a, b, and c,

$$\text{If } a < b, \text{ then } a+c < b+c$$
$$\text{If } a > b, \text{ then } a+c > b+c$$

The Addition Property states that the sense, or direction, of an inequality remains unchanged if the same number is added to each side.

Multiplication Properties for Inequalities: For real numbers a, b, and c,

$$\text{If } a < b, \text{ and if } c > 0, \text{ then } ac < bc$$
$$\text{If } a < b, \text{ and if } c < 0, \text{ then } ac > bc$$

$$\text{If } a > b, \text{ and if } c > 0, \text{ then } ac > bc$$
$$\text{If } a > b, \text{ and if } c < 0, \text{ then } ac < bc$$

The Multiplication Properties state that the sense, or direction, of an inequality remains the same if each side is multiplied by a positive real number; whereas the direction is reversed if each side is multiplied by a negative real number.

Example 3:

a) If $3x < 12$, then $\frac{1}{3}(3x) < \frac{1}{3}(12)$ or $x < 4$

b) If $\frac{x}{-4} > 20$, then $-4\left(\frac{x}{-4}\right) < -4(20)$ or $x < -80$

c) If $-5x < -15$ then $\frac{-5x}{-5} > \frac{-15}{-5}$ or $x > 3$

d) If $-x > 9$, then $-1(-x) < -1(9)$ or $x < -9$

Appendix A.9 – Interval Notation; Solving Inequalities (continued)

Reciprocal Property for Inequalities:

If $a > 0$, then $\frac{1}{a} > 0$ If $\frac{1}{a} > 0$, then $a > 0$

If $a < 0$, then $\frac{1}{a} < 0$ If $\frac{1}{a} < 0$, then $a < 0$

The Reciprocal Property states that the reciprocal of a positive real number is positive and that the reciprocal of a negative real number is negative.

Solve Inequalities

An **inequality in one variable** is a statement involving two expressions, at least one containing the variable, separated by one of the inequality symbols, $<, \le, >,$ or $\ge$. To solve an inequality means to find all values of the variable for which the statement is true. These values are called **solutions** of the inequality.

For example, the following are all inequalities involving one variable x:

$x+4<9$ $3x-2\ge 6$ $x^2-2\le 10$ $\frac{x+2}{x-3}>0$

As with equations, one method for solving an inequality is to replace it by a series of equivalent inequalities until an inequality with an obvious solution, such as $x<3$, is obtained. Equivalent inequalities are obtained by applying some of the same properties that are used to find equivalent equations. The addition property and the multiplication properties for inequalities form the basis for the following procedures.

Procedures that Leave the Inequality Symbol Unchanged

1. Simplify both sides of the inequality by combining like terms and eliminating parentheses:
   Replace $x+2+6>2x+4(x+1)$
   By $x+8>2x+4x+4$
   $x+8>6x+4$

2. Add or subtract the same expression on both sides of the inequality:
   Replace $3x-7<6$
   By $(3x-7)+7<6+7$

3. Multiply or divide both sides of the inequality by the same *positive* expression:
   Replace $5x>15$
   By $\frac{5x}{5}>\frac{15}{5}$

Procedures that Reverse the Sense or Direction of the Inequality Symbol

1. Interchange the two sides of the inequality:
   Replace $4<x$
   By $x>4$

2. Multiply or divide both sides of the inequality by the same *negative* expression:
   Replace $-3x>12$
   By $\frac{-3x}{-3}<\frac{12}{-3}$

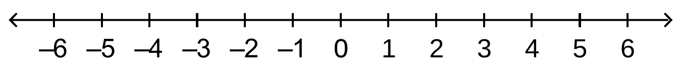
You solve inequalities using many of the same steps that you would use to solve equations. In writing the solution of an inequality, either set notation or interval notation may be used, whichever is more convenient.

Appendix A.9 – Interval Notation; Solving Inequalities (continued)

Example 4: Solve $3x-1>3+x$. Graph the solution set.

$$3x-1>3+x$$
$$3x-x-1>3+x-x$$
$$2x-1>3$$
$$2x-1+1>3+1$$
$$2x>4$$
$$\frac{2x}{2}>\frac{4}{2}$$
$$x>2$$

Solution: $\{x|x>2\}$ or $(2,\infty)$

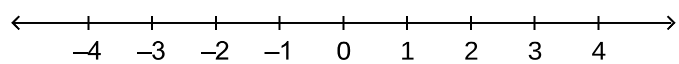


Solve Combined Inequalities

Example 5: Solve $-5\le 4-3x\le 2$. Graph the solution set.

$$-5\le 4-3x\le 2$$
$$-5-4\le 4-4-3x\le 2-4$$
$$-9\le -3x\le -2$$
$$\frac{-9}{-3}\ge\frac{-3x}{-3}\ge\frac{-2}{-3}$$
$$3\ge x\ge\frac{2}{3}\quad\Rightarrow\quad\frac{2}{3}\le x\le 3$$

Solution: $\left\{x\,\middle|\,\frac{2}{3}\le x\le 3\right\}$ or $\left[\frac{2}{3},3\right]$



The inequality in Example 5, $-5\le 4-3x\le 2$, is equivalent to the two inequalities $-5\le 4-3x$ and $4-3x\le 2$. You could also solve Example 5 by solving each of these two inequalities separately, and then combining the solution sets for each in a more compact form.

Example 5a: Solve $-5\le 4-3x\le 2$.

$-5\le 4-3x$	and	$4-3x\le 2$
$-5-4\le 4-4-3x$		$4-4-3x\le 2-4$
$-9\le -3x$		$-3x\le -2$
$\frac{-9}{-3}\ge\frac{-3x}{-3}$		$\frac{-3x}{-3}\ge\frac{-2}{-3}$
$3\ge x$		$x\ge\frac{2}{3}$

The solution set of the original pair of inequalities consists of all x for which $3\ge x$ and $x\ge\frac{2}{3}$. This may be written more compactly as $\left\{x\,\middle|\,\frac{2}{3}\le x\le 3\right\}$ or $\left[\frac{2}{3},3\right]$, which is the same solution as Example 5 above.

Appendix A.9 – Interval Notation; Solving Inequalities (continued)

Using the Reciprocal Property to Solve an Inequality

Example 6: Solve $(5x-2)^{-1} > 0$. Graph the solution set.

Recall that $(5x-2)^{-1} = \frac{1}{5x-2}$. The Reciprocal Property states that if $\frac{1}{a} > 0$, then $a > 0$.

$(5x-2)^{-1} > 0$

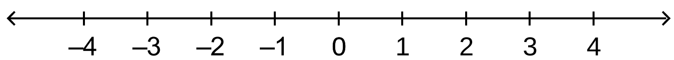
$\frac{1}{5x-2} > 0$

$5x-2 > 0$ by the Reciprocal Property

$5x > 2$

$x > \frac{2}{5}$

The solution set is $\left\{x \mid x > \frac{2}{5}\right\}$, that is, all x in the interval $\left(\frac{2}{5}, \infty\right)$.



Solve Inequalities Involving Absolute Value

Recall that the absolute value of a number A, $|A|$, is the distance on a number line from the point whose coordinate is A to the origin.

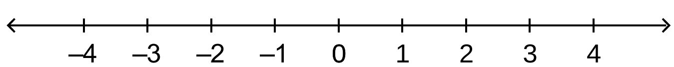
The **absolute value** of a real number *a*, denoted by the symbol $|a|$, is defined by the rules

$|a| = a$ if $a \geq 0$ and $|a| = -a$ if $a < 0$.

Examples: $|7| = 7$ and $|-12| = 12$

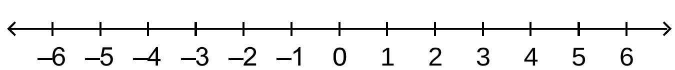
Example 7: Solve $|x| < 3$. Graph the solution set.

You want all points whose coordinate x is a distance less than 3 units from the origin. Because any x between -3 and 3 satisfies the condition $|x| < 3$, the solution set consists of all numbers x for which $-3 < x < 3$, that is, all x in the interval $(-3, 3)$.



Example 8: Solve $|x| > 4$. Graph the solution set.

You want all points whose coordinate x is a distance greater than 4 units from the origin. Any number x less than -4 or greater than 4 satisfies the condition $|x| > 4$. The solution set consists of all numbers x for which $x < -4$ or $x > 4$, that is, all x in $(-\infty, -4) \cup (4, \infty)$.



Appendix A.9 – Interval Notation; Solving Inequalities (continued)

Examples 7 and 8 illustrate the following:

Theorem: If *a* is any positive number then,

$|x| < a$ is equivalent to $-a < x < a$

$|x| \le a$ is equivalent to $-a \le x \le a$

$|x| > a$ is equivalent to $x < -a$ or $x > a$

$|x| \ge a$ is equivalent to $x \le -a$ or $x \ge a$

Example 9: Solve $|3x+5| \le 8$. Graph the solution set.

$-8 \le 3x+5 \le 8$ by the above theorem

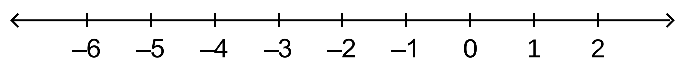
$-8-5 \le 3x+5-5 \le 8-5$

$-13 \le 3x \le 3$

$$\frac{-13}{3} \le \frac{3x}{3} \le \frac{3}{3}$$

$$\frac{-13}{3} \le x \le 1$$

The solution set is $\left\{ x \,\middle|\, \frac{-13}{3} \le x \le 1 \right\}$.



Example 10: Solve $|2x-3| > 5$. Graph the solution set.

$2x-3 < -5$ or $2x-3 > 5$ by the above theorem

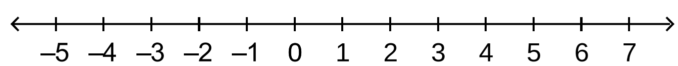
$2x-3+3 < -5+3$ $\quad$ $2x-3+3 > 5+3$

$2x < -2$ $\quad$ $2x > 8$

$\frac{2x}{2} < \frac{-2}{2}$ $\quad$ $\frac{2x}{2} > \frac{8}{2}$

$x < -1$ or $x > 4$

The solution set is $\{x \mid x < -1 \text{ or } x > 4\}$



All material has been taken from ***Precalculus,*** by M. Sullivan, 10th Edition