Appendix A.10 – nth Roots; Rational Exponents, Day1

Work with n^{th} Roots

The principal n^{th} root of a real number *a*, $n \ge 2$ an integer, symbolized by $\sqrt[n]{a}$, is defined as follows:

$\sqrt[n]{a} = b$ means $a = b^n$ where $a \ge 0$ and $b \ge 0$ if *n* is even and *a, b* are any real numbers if *n* is odd.

Notice that if *a* is negative and *n* is even, then $\sqrt[n]{a}$ is not defined. When it is defined, the principal n^{th} root of a number is unique.

The symbol $\sqrt[n]{a}$ for the principal n^{th} root of *a* is called a **radical**; the integer *n* is called the **index**, and *a* is called the **radicand**. If the index of a radical is 2, we call $\sqrt[2]{a}$ the **square root** of *a* and omit the index 2 by simply writing $\sqrt{a}$. If the index is 3, we call $\sqrt[3]{a}$ the **cube root** of *a*.

The symbol $\sqrt[n]{a}$ means "give me the number that, when raised to the power n, equals a."

Example 1: Simplifying principal n^{th} roots

a) $\sqrt[3]{8} = \sqrt[3]{2^3}$
$= 2$

b) $\sqrt[3]{-64} = \sqrt[3]{(-4)^3}$
$= -4$

c) $\sqrt[4]{\frac{1}{16}} = \sqrt[4]{\left(\frac{1}{2}\right)^4}$
$= \frac{1}{2}$

d) $\sqrt[6]{(-4)^6} = |-2|$
$= 2$

These are examples of **perfect roots**, since each simplifies to a rational number. Notice the absolute value in Example 1 (d). If *n* is even, then the principal n^{th} root must be nonnegative.

In general, if $n \ge 2$ is an integer and *a* is a real number, you have

$\sqrt[n]{a^n} = a$ if $n \ge 3$ is odd

$\sqrt[n]{a^n} = |a|$ if $n \ge 2$ is even

Radicals provide a way of representing many irrational real numbers. For example, it can be shown that there is no rational number whose square is 2. Using radicals, you can say that $\sqrt{2}$ is the positive number whose square is 2.

You can use your calculator to approximate roots.

Example 2: Use a calculator to approximate $\sqrt[6]{23}$.

On a TI-83 there are two ways to calculate this root.
1) Type in 23, the caret key ^, (1/6), Enter
2) Type in 6, the MATH key, select 5: $\sqrt[x]{\ }$, 23, Enter

$\sqrt[6]{23} \approx 1.686376$

$\sqrt[6]{23} \approx 1.69$

Simplify Radicals

Let $n \ge 2$ and $m \ge 2$ denote integers, and let *a* and *b* represent real numbers. Assuming that all radicals are defined, we have the following properties:

Property of Radicals

$\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$ $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ $b \ne 0$ $\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

When used in reference to radicals, the direction to "simplify" will mean to remove from the radicals any perfect roots that occur as factors.

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Example 3: Simplify.

a) $\sqrt{50}$

$$\sqrt{50} = \sqrt{25(2)} = \sqrt{25}\sqrt{2} = 5\sqrt{2}$$

b) $\sqrt[3]{40}$

$$\sqrt[3]{40} = \sqrt[3]{8(5)} = \sqrt[3]{8}\,\sqrt[3]{5} = 2\,\sqrt[3]{5}$$

c) $\sqrt[3]{-81x^4}$

$$\sqrt[3]{-81x^4} = \sqrt[3]{-27x^3(3x)} = \sqrt[3]{-27x^3}\,\sqrt[3]{3x} = -3x\,\sqrt[3]{3x}$$

d) $\sqrt[4]{\dfrac{81x^5}{256}}$

$$\sqrt[4]{\frac{81x^5}{256}} = \sqrt[4]{\frac{3^4x^4x}{4^4}} = \sqrt[4]{\left(\frac{3x}{4}\right)^4(x)} = \sqrt[4]{\left(\frac{3x}{4}\right)^4}\left(\sqrt[4]{x}\right) = \left|\frac{3x}{4}\right|\left(\sqrt[4]{x}\right)$$

Two or more radicands can be combined, provided that they have the same index and the same radicand. Such radicals are called **like radicals**.

Example 4: Simplify

a) $$-6\sqrt{20} + \sqrt{5} = -6\sqrt{4(5)} + \sqrt{5} = -6\sqrt{4}\sqrt{5} + \sqrt{5} = -6(2)\sqrt{5} + \sqrt{5} = -12\sqrt{5} + \sqrt{5} = -11\sqrt{5}$$

b) $$\sqrt[3]{27x^4} + \sqrt[3]{-x} = \sqrt[3]{3^3x^3x} + \sqrt[3]{-x} = \sqrt[3]{(3x)^3}\left(\sqrt[3]{x}\right) + \sqrt[3]{-1}\left(\sqrt[3]{x}\right) = 3x\left(\sqrt[3]{x}\right) - 1\left(\sqrt[3]{x}\right) = (3x-1)\left(\sqrt[3]{x}\right)$$

Rationalize Denominators

When radicals occur in quotients, you rewrite the quotient so that the new denominator contains no radicals. This process is referred to as **rationalizing the denominator**.

The idea is to multiply by an appropriate expression so that the new denominator contains no radicals. For example:

If a Denominator Contains the Factor	**Multiply by**	**To Obtain a Denominator Free of Radicals**
$\sqrt{3}$	$\sqrt{3}$	$\left(\sqrt{3}\right)^2 = 3$
$\sqrt{3}+1$	$\sqrt{3}-1$	$\left(\sqrt{3}\right)^2 - 1^2 = 3-1 = 2$
$\sqrt{2}-5$	$\sqrt{2}+5$	$\left(\sqrt{2}\right)^2 - 5^2 = 2-25 = -23$
$\sqrt{7}-\sqrt{5}$	$\sqrt{7}+\sqrt{5}$	$\left(\sqrt{7}\right)^2 - \left(\sqrt{5}\right)^2 = 7-5 = 2$
$\sqrt[3]{9}$	$\sqrt[3]{3}$	$\sqrt[3]{9}\left(\sqrt[3]{3}\right) = \sqrt[3]{27} = 3$

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Example 5: Rationalize the denominator of each expression

a) $\frac{6}{\sqrt{2}} = \frac{6}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)$

$= \frac{6\sqrt{2}}{2}$

$= 3\sqrt{2}$

b) $\frac{\sqrt{7}}{\sqrt[3]{3}} = \frac{\sqrt{7}}{\sqrt[3]{3}}\left(\frac{\sqrt[3]{9}}{\sqrt[3]{9}}\right)$

$= \frac{\sqrt{7}\sqrt[3]{9}}{\sqrt[3]{27}}$

$= \frac{\sqrt{7}\sqrt[3]{9}}{3}$

c) $\frac{\sqrt{x}+3}{\sqrt{x}-3} = \frac{\sqrt{x}+3}{\sqrt{x}-3}\left(\frac{\sqrt{x}+3}{\sqrt{x}+3}\right)$

$= \frac{\left(\sqrt{x}+3\right)^2}{\left(\sqrt{x}\right)^2 - 3^2}$

$= \frac{\left(\sqrt{x}\right)^2 + 6\sqrt{x} + 9}{x-9}$

$= \frac{x + 6\sqrt{x} + 9}{x-9}$

Solve Radical Equations

When the variable in an equation occurs in a square root, cube root, and so on – that is, when it occurs in a radical – the equation is called a **radical equation**. Sometimes a suitable operation will change a radical equation to one that is linear or quadratic. A commonly used procedure is to isolate the most complicated radical on one side of the equation and then eliminate it by raising each side to a power equal to the index of the radical. Care must be taken, however, because apparent solutions that are not, in fact, solutions of the original equation may result. These are called **extraneous solutions**. Therefore, you need to check all answers when working with radical equations, and you check them in the original equation.

Example 6: Find the real solutions of the equation: $\sqrt[5]{2x-6} - 4 = 0$

The equation contains a radical whose index is 5. Isolate it on the left side.

$\sqrt[5]{2x-6} = 4$

Now raise each side to the fifth power (the index of the radical is 5) and solve.

$\left(\sqrt[5]{2x-6}\right)^5 = 4^5$

$2x - 6 = 1024$

$2x = 1030$

$x = 515$

Check: $\sqrt[5]{2x-6} - 4 = 0$, set $x = 515$

$\sqrt[5]{2(515)-6} - 4 = 0$

$\sqrt[5]{1030-6} - 4 = 0$

$\sqrt[5]{1024} - 4 = 0$

$4 - 4 = 0$

$0 = 0$ True

So, the solution set is $\{515\}$.

All material has been taken from *Precalculus,* by M. Sullivan, 10th Edition