Appendix A.3 – Polynomials

We have described algebra as a generalization of arithmetic in which letters are used to represent real numbers. From now on, we shall use the letters at the end of the alphabet, such as x, y, and z, to represent variables and use the letters at the beginning of the alphabet, such as a, b, and c, to represent constants. In the expressions $3x+5$ and $ax+b$, it is understood that x is a variable and that a and b are constants, even though the constants a and b are unspecified. As you will find out, the context usually makes the intended meaning clear.

Recognize Monomials

A monomial in one variable is the product of a constant and a variable raised to a nonnegative integer power. A monomial is of the form ax^k, where *a* is a constant, *x* is a variable, and $k \geq 0$ is an integer. The constant *a* is called coefficient of the monomial. If $a \neq 0$, then k is called the degree of the monomial.

Example 1: Examples of Monomials

Monomial	Coefficient	Degree
a) $5x^2$	5	2
b) $-\sqrt{7}x^3$	$-\sqrt{7}$	3
c) 4	4	0
d) $-6x$	-6	1
e) x^4	1	4

Example 2: Examples of Nonmonomial Expressions

a) $4x^{\left(\frac{1}{3}\right)}$ is not a monomial, since the exponent of the variable x is $\frac{1}{3}$ and $\frac{1}{3}$ is not a nonnegative integer.

b) $6x^{-4}$ is not monomial, since the exponent of the variable x is -4 and -4 is not a nonnegative integer.

Recognize Polynomials

Two monomials with the same variable raised to the same power are called **like terms**. For example, $5x^3$ and $-8x^3$ are like terms. In contrast, the monomials $2x^4$ and $9x^2$ are not like terms, since their powers are not the same.

You can add and subtract like terms using the Distributive Property.

For example, $3x^2+5x^2=(3+5)x^2 = 8x^2$ and $9x^4-3x^4=(9-3)x^4 = 6x^4$

The sum or difference of two monomials having different degrees is called a **binomial**. The sum or difference of three monomials with three different degrees is called a **trinomia**l.

For example, x^3-7 is a binomial.
x^2+4x-7 is a trinomial.
$4x^2+6x^2-5=10x^2-5$ is a binomial.

Definition: A **polynomial** in one variable is an algebraic expression of the form $a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$, where $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants, called the **coefficients** of the polynomial, $n \geq 0$ is an integer, and *x* is a variable.
If $a_n \neq 0$, it is called the **leading coefficient**, a_nx^n is called the **leading term**, and *n* is the **degree** of the polynomial.

So, a polynomial is a sum of monomials.

Appendix A.3 – Polynomials (continued)

The monomials that make up a polynomial are called its **terms**. If all of the coefficients are 0, the polynomial is called the **zero polynomial**, which has no degree.

Polynomials are usually written in **standard form**, beginning with the nonzero term of highest degree and continuing with terms in descending order according to degree. If a power of *x* is missing, it is because its coefficient is zero.

Example 3: Examples of Polynomials

	Polynomial	Coefficients	Degree
a)	$7x^4 - 5x^2 + x - 6$	$7, -5, 1, -6$	4
b)	$4x^2 - 7 = 4x^2 + 0x + (-7)$	$4, 0, -7$	2
c)	$6 - 5x + 2x^2 = 2x^2 - 5x + 6$	$2, -5, 6$	2
d)	$8x + \sqrt{6}$	$8, \sqrt{6}$	1
e)	9	9	0
f)	0	0	No degree

Although we have been using *x* to represent the variable, letters such as *y* and *z* are also commonly used.

$4x^3 - 5x + 3$ is a polynomial (in x) of degree 3.

$8y^4 - 6y^3 + 2y - 5$ is a polynomial (in y) of degree 4.

$z^5 + 3z^2 + 2\pi$ is a polynomial (in z) of degree 5.

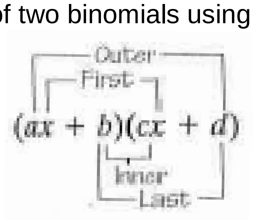
Algebraic expressions such as $\frac{2}{x}$ and $\frac{x^2+4}{x-7}$ are not polynomials.

$\frac{2}{x}$ is not a polynomial because $\frac{2}{x} = 2x^{-1}$ has an exponent that is not a nonnegative integer.

The second expression is a quotient of two polynomials, but the polynomial in the denominator has degree greater than 0, so the expression cannot be a polynomial.

Know Formulas for Special Products

Certain products, which we call **special products**, occur frequently in algebra. For example, you can find the product of two binomials using the **FOIL** (**F**irst, **O**uter, **I**nner, **L**ast) method.



$$= ax(cx+d) + b(cx+d)$$

$$= \overbrace{ax \cdot cx}^{\text{First}} + \overbrace{ax \cdot d}^{\text{Outer}} + \overbrace{b \cdot cx}^{\text{Inner}} + \overbrace{b \cdot d}^{\text{Last}}$$
$$= acx^2 + adx + bcx + bd$$
$$= acx^2 + (ad + bc)x + bd$$

Example 4: Using FOIL

a) $(x+4)(x-4) = x^2 - 4x + 4x - 16$
$= x^2 - 16$

b) $(2x-3)(4x-5) = 8x^2 - 10x - 12x + 15$
$= 8x^2 - 22x + 15$

c) $(3x+7)^2 = (3x+7)(3x+7)$
$= 9x^2 + 21x + 21x + 49$
$= 9x^2 + 42x + 49$

Appendix A.3 – Polynomials (continued)

Some products have been given special names because of their form. In the list that follows, *x* and *a* are real numbers.

Difference of Two Squares: $x^2 - a^2 = (x+a)(x-a)$

Squares of Binomials, or Perfect Squares: $(x+a)^2 = x^2 + 2ax + a^2$

$(x-a)^2 = x^2 - 2ax + a^2$

Cubes of Binomials, or Perfect Cubes: $(x+a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$

$(x-a)^3 = x^3 - 3ax^2 + 3a^2x - a^3$

Difference of Two Cubes: $x^3 - a^3 = (x-a)(x^2 + ax + a^2)$

Sum of Two Cubes: $x^3 + a^3 = (x+a)(x^2 - ax + a^2)$

Divide Polynomials Using Long Division

The procedure for dividing two polynomials is similar to the procedure for dividing two integers.

Divide 446 by 13.

Quotient → 34

Divisor → 13) 446 ← Dividend

-39

56

-52

4 ← Remainder

So, $\frac{446}{13} = 34 + \frac{4}{13}$

In the long-division process, the number 13 is called the **divisor**, the number 446 is called the **dividend**, the number 34 is called the **quotient**, and the number 4 is called the **remainder**.

To check the answer obtained in a division problem, multiply the quotient by the divisor and add the remainder. The answer should be the dividend.

$$(\text{Quotient})(\text{Divisor}) + \text{Remainder} = \text{Dividend}$$

To divide two polynomials, the polynomials must be in standard form. The process then follows a pattern similar to the division problem above.

Example 5: Use Long Division to divide $x^3 + 3x^2 - 4x - 14$ by $x - 2$.

Quotient → $x^2 + 5x + 6$

Divisor → $x - 2\,)\,x^3 + 3x^2 - 4x - 14$ ← Dividend

$-(x^3 - 2x^2)$

$5x^2 - 4x$

$-(5x^2 - 10x)$

$6x - 14$

$-(6x - 12)$

-2 ← Remainder

Appendix A.3 – Polynomials (continued)

Check:

$$\begin{aligned}(\text{Quotient})(\text{Divisor}) + \text{Remainder} &= (x^2+5x+6)(x-2) + -2\\ &= (x^3+5x^2+6x) - (2x^2+10x+12) - 2\\ &= x^3+5x^2+6x - 2x^2-10x-12-2\\ &= x^3+3x^2-4x-14\\ &= \text{Dividend}\end{aligned}$$

Thus, $\dfrac{x^3+3x^2-4x-14}{x-2} = x^2+5x+6-\dfrac{2}{x-2}$

If the dividend, say $4x^3+7x^2+5$, is missing a term (here the x term is missing), you must put a placeholder in the dividend for the missing term (here put in 0x).

The process of dividing two polynomials leads to the following theorem.

Theorem: Let Q be a polynomial of positive degree, and let P be a polynomial whose degree is greater than or equal to the degree of Q. The remainder after dividing P by Q is either the zero polynomial or a polynomial whose degree is less than the degree of the divisor Q.

Factor Polynomials

Consider the following product: $(2x-3)(x+4)=2x^2+5x-12$

The two polynomials on the left side are called f**actors** of the polynomial on the right side. Expressing a given polynomial as a product of other polynomials, that is, finding the factors of a polynomial, is called **factoring**.

We will restrict our discussion here to factoring polynomials in one variable into products of polynomials in one variable, where all coefficients are integers. We call this **factoring over the integers**.

Any polynomial can be written as the product of 1 times itself or as -1 times its additive inverse. If a polynomial cannot be written as the product of two other polynomials (excluding 1 and -1), then the polynomial is said to be prime. When a polynomial has been written as a product consisting only of prime factors, it is said to be **factored completely**. Examples of prime polynomials (over the integers) are

$$2 \quad 3 \quad 7 \quad x \quad x-1 \quad x+2 \quad 3x-5 \quad x^2+3$$

The first factor to look for in a factoring problem is a common monomial factor present in each term of the polynomial. If one is present, use the Distributive Property to factor it out.

Example 6: Identifying Common Monomial Factors

Polynomial	Common Monomial Factor	Remaining Factor	Factored Form
$2x+4$	2	$x+2$	$2x+4=2(x+2)$
$3x-6$	3	$x-2$	$3x-6=3(x-2)$
$2x^2-4x+8$	2	x^2-2x+4	$2x^2-4x+8=2(x^2-2x+4)$
$8x-12$	4	$2x-3$	$8x-12=4(2x-3)$
x^2+x	x	$x+1$	$x^2+x=x(x+1)$
x^3-3x^2	x^2	$x-3$	$x^3-3x^2=x^2(x-3)$
$6x^2+9x$	$3x$	$2x+3$	$6x^2+9x=3x(2x+3)$

Notice that, once all common monomial factors have been removed from a polynomial, the remaining factor is either a prime polynomial of degree 1 or a polynomial of degree 2 or higher.

Appendix A.3 – Polynomials (continued)

The following example illustrates several factoring techniques.

Example 7: Factoring Polynomials

Factor each polynomial completely.

a) $x^4 - 16$ b) $x^3 - 1$ c) $9x^2 - 6x + 1$

d) $x^2 + 4x - 12$ e) $3x^2 + 10x - 8$ f) $x^3 - 4x^2 + 2x - 8$

Solution: a) $x^4 - 16 = (x^2+4)(x^2-4)$ Difference of squares

$= (x^2+4)(x+2)(x-2)$ Difference of squares

b) $x^3 - 1 = (x-1)(x^2+x+1)$ Difference of cubes

c) $9x^2 - 6x + 1 = (3x-1)^2$ Perfect square

d) $x^2 + 4x - 12 = (x+6)(x-2)$ The product of 6 and -2 is $-12,$ and the sum of 6 and -2 is 4

e) $3x^2 + 10x - 8 = (3x-2)(x+4)$

f) $x^3 - 4x^2 + 2x - 8 = (x^3 - 4x^2) + (2x - 8)$ Group terms Known as factoring by grouping

$= x^2(x - 4x) + 2(x - 4)$ Distributive Property

$= (x^2 + 2)(x - 4)$ Distributive Property

Complete the Square

The idea behind completing the square in one variable is to "adjust" an expression of the form $x^2 + bx$ to make it a perfect square. Perfect squares are trinomials of the form $x^2 + 2ax + a^2 = (x+a)^2$ or $x^2 - 2ax + a^2 = (x-a)^2$. For example, $x^2 + 6x + 9$ is a perfect square because $x^2 + 6x + 9 = (x+3)^2$ and $p^2 - 12p + 36$ is a perfect square because $p^2 - 12p + 36 = (p-6)^2$.

So, how do you "adjust" $x^2 + bx$ to make it a perfect square? You do so by adding a number.

Completing the Square: To complete the square of the expression $x^2 + bx,$ add the square of half of the coefficient of x, that is, add $\left(\frac{b}{2}\right)^2$.

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

Example 8: Find the term that should be added to the expression to create a perfect square trinomial (perfect square).

a) $x^2 - 8x$

$x^2 + -8x$

$$\left(\frac{b}{2}\right)^2 = \left(\frac{-8}{2}\right)^2$$
$$= (-4)^2$$
$$= 16$$

$$x^2 - 8x + \left(\frac{-8}{2}\right)^2 = x^2 - 8x + (-4)^2$$
$$= x^2 - 8x + 16$$
$$= (x-4)^2$$

b) $x^2 + 7x$

$$\left(\frac{b}{2}\right)^2 = \left(\frac{7}{2}\right)^2$$
$$= \frac{49}{4}$$

$$x^2 + 7x + \left(\frac{7}{2}\right)^2 = x^2 + 7x + \frac{49}{4}$$
$$= \left(x + \frac{7}{2}\right)^2$$

Appendix A.3 – Polynomials (continued)

When completing the square, you need the leading coefficient *a* to be 1. If it is not 1, factor out *a* before completing the square.

Example 9: Solve $x^2 + 10x = 24$ by completing the square.

$$x^2 + 10x = 24$$
$$\left(\frac{b}{2}\right)^2 = \left(\frac{10}{2}\right)^2$$
$$= 5^2$$
$$= 25$$
$$x^2 + 10x + 25 = 24 + 25$$
$$(x+5)^2 = 49$$
$$x + 5 = \pm\sqrt{49}$$
$$x + 5 = \pm 7$$
$$x = -5 \pm 7$$
$$x = -5 + 7 \quad \text{or} \quad x = -5 - 7$$
$$x = 2 \quad \text{or} \quad x = -12$$

Example 10: Solve $3x^2 - 6x - 9 = 0$ by completing the square.

$$3x^2 - 6x = 9$$
$$3(x^2 - 2x) = 9$$
$$\left(\frac{b}{2}\right)^2 = \left(\frac{-2}{2}\right)^2$$
$$= (-1)^2$$
$$= 1$$
$$3(x^2 - 2x + 1) = 9 + 3$$
$$3(x-1)^2 = 12$$
$$(x-1)^2 = 4$$
$$x - 1 = \pm\sqrt{4}$$
$$x - 1 = \pm 2$$
$$x = 1 \pm 2$$
$$x = 1 + 2 \quad \text{or} \quad x = 1 - 2$$
$$x = 3 \quad \text{or} \quad x = -1$$