Appendix A7 – Complex Numbers; Quadratic Equations in the Complex Number System

The square root of a real number is nonnegative, so there is no real number x for which $x^2 = -1$. To remedy this, a number called the imaginary unit was introduced (by Euler).

The **imaginary unit**, which we denote by i, is the number whose whose square is -1. That is $i^2 = -1$. The number system that results from introducing i is the complex number system.

$$i = \sqrt{-1} \qquad i^2 = -1$$

$$i^3 = i^2(i) = (-1)\,i = -i \qquad i^5 = i^4(i) = (1)i = i \qquad i^7 = i^4(i^3) = (1)i^3 = -i$$

$$i^4 = i^2(i^2) = (-1)(-1) = 1 \qquad i^6 = i^4(i^2) = (1)(-1) = -1 \qquad i^8 = i^4(i^4) = (1)(1) = 1$$

$\Rightarrow$ The powers of i are cyclic and repeat in a pattern of four numbers.

So,
$$\begin{aligned} i^{66} &= i^{4(16)+2} \\ &= i^{4(16)} i^2 \\ &= \left(i^4\right)^{16} i^2 \\ &= (1)^{16}(-1) \\ &= (1)(-1) \\ &= -1 \end{aligned}$$

OR i^{66} : $66 \div 4 = 16$, $66 - 64 = 2$ So, $i^{66} \equiv i^2 = -1$

Complex Numbers

A complex number is any number that can be written in the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$. The real number a is called the real part of the number $a + bi$, and the real number b is called the imaginary part of the number $a + bi$.

The form $a + bi$ is called the standard form of a complex number. Real numbers are complex numbers for which $b = 0$. So, we write $a + 0i$ as a. A complex number is called a pure imaginary number if its real part, a, is 0. So, $0 + bi$, written as bi, is a pure imaginary number.

Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal.
$\Rightarrow$ $a + bi = c + di$ if and only if $a = c$ and $b = d$.

Add, Subtract, Multiply and Divide Complex Numbers

Addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$

Subtraction: $(a + bi) - (c + di) = (a - c) + (b - d)i$

Multiplication:

$$\begin{aligned} (a + bi)(c + di) &= ac + adi + bci + bdi^2 \quad \text{using FOIL} \\ &= ac + adi + bci + bd(-1) \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

(There is no need to memorize this, just follow the usual rules for multiplying two binomials)

Example 1:
$$\begin{aligned} (4 + 6i) + (-1 + 5i) &= (4 + -1) + (6 + 5)i \\ &= 3 + 11i \end{aligned}$$

$$\begin{aligned} (3 + 7i) - (6 + 2i) &= (3 - 6) + (7 - 2)i \\ &= -3 + 5i \end{aligned}$$

$$\begin{aligned} (2 + 3i)(4 + 5i) &= 2(4) + 2(5i) + 3i(4) + 3i(5i) \\ &= 8 + 10i + 12i + 15i^2 \\ &= 8 + 22i + 15(-1) \\ &= 8 + 22i - 15 \\ &= -7 + 22i \end{aligned}$$

Appendix A7 – Complex Numbers; Quadratic Equations in the Complex Number System (continued)

The conjugate of a complex number $a+bi$ is $a-bi$. The conjugate of $a+bi$ is denoted by $\overline{a+bi}$.

Example 2: $\overline{2+5i}=2-5i$ $\quad\quad \overline{-1-3i}=-1+3i$

Theorem: The product of a complex number and its conjugate is a nonnegative real number.
That is, if $z=a+bi$, then $z\bar{z}=a^2+b^2$.

Example 3: $z=2+3i$, $\overline{z}=2-3i$

$$\begin{aligned} z\overline{z} &= (2+3i)(2-3i) \\ &= 4-6i+6i-9i^2 \\ &= 4-9(-1) \\ &= 13 \end{aligned} \quad \text{OR} \quad \begin{aligned} z\overline{z} &= 2^2+3^2 \\ &= 4+9 \\ &= 13 \end{aligned}$$

To simplify a quotient with an imaginary number in the denominator, multiply by a fraction equal to 1, using the conjugate of the denominator. In other words, multiply the numerator and the denominator of the quotient by the conjugate of the denominator. This process is called rationalizing the denominator.

Example 4: Simplify $\frac{3+5i}{2-3i}$.

$$\begin{aligned} \frac{3+5i}{2-3i} &= \frac{3+5i}{2-3i}\left(\frac{2+3i}{2+3i}\right) \\ &= \frac{6+9i+10i+15(i^2)}{4+6i-6i-9(i^2)} \\ &= \frac{6+19i+15(-1)}{4-9(-1)} \\ &= \frac{-9+19i}{13} \\ &= \frac{-9}{13}+\frac{19}{13}i \end{aligned}$$

Never leave an imaginary number in a denominator – always rationalize the denominator.

Theorem – The conjugate of a real number is the real number itself.

Example 5:
$$\begin{aligned} \overline{3} &= \overline{3+0i} \\ &= 3-0i \\ &= 3 \end{aligned}$$

Theorem – The conjugate of the conjugate of a complex number is the complex number itself: $\overline{\overline{z}}=z$.

The conjugate of the sum of two complex numbers equals the sum of their conjugates: $\overline{z+w}=\overline{z}+\overline{w}$.

The conjugate of the product of two complex numbers equals the product of their conjugates:
$\overline{(z)(w)}=\overline{z}\left(\overline{w}\right)$.

Appendix A7 – Complex Numbers; Quadratic Equations in the Complex Number System (continued)

Recall the special product formula: $(x+a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$

Example 6: Simplify $(3+i)^3$.

$$\begin{aligned}(3+i)^3 &= 3^3 + 3(i)(3^2) + 3(i^2)(3) + i^3\\ &= 27 + 3(9)i + 9(-1) + -i\\ &= 27 + 27i + 9(-1) + -i\\ &= 27 - 9 + 27i + -i\\ &= 18 + 26i\end{aligned}$$

OR

$$\begin{aligned}(3+i)^3 &= (3+i)^2(3+i)\\ &= (9+3i+3i+i^2)(3+i)\\ &= (9+6i+-1)(3+i)\\ &= (8+6i)(3+i)\\ &= (24+8i+18i+6i^2)\\ &= (24+26i+6(-1))\\ &= (24+26i+-6)\\ &= 18+26i\end{aligned}$$

Solve Quadratic Equations in the Complex Number System

Quadratic equations with a negative discriminant have no real number solution. However, if we extend our number system to allow complex numbers, quadratic equations will always have a solution. Since the solution to a quadratic equation involves the square root of the discriminant, we begin with a discussion of square roots of negative numbers.

Definition: If N is a positive real number, we define the principal square root of $-N$, denoted by $\sqrt{-N}$, as $\sqrt{-N} = \sqrt{N}\, i$, where i is the imaginary unit and $i^2 = -1$.

Example 7: $\sqrt{-3} = \sqrt{3}\, i$

$$\begin{aligned}\sqrt{-12} &= \sqrt{12}\, i\\ &= \sqrt{4}\sqrt{3}\, i\\ &= 2\sqrt{3}\, i\end{aligned}$$

Example 8: Solve the equation $x^2 + 16 = 0$.

$$\begin{aligned}x^2 + 16 &= 0\\ x^2 &= -16\\ x &= \pm\sqrt{-16}\\ &= \pm\sqrt{16}\, i\\ &= \pm 4i\end{aligned}$$

Warning: When working with square roots of negative numbers, do not set the square root of a product equal to the product of the square roots (which can be done with positive real numbers).

Example 9:

$$\begin{aligned}8 &= \sqrt{64}\\ &= \sqrt{-16(-4)}\\ &= \sqrt{-16}\sqrt{-4} \quad \text{This is where the error occurs } (\neq)\\ &= \sqrt{16}\, i\sqrt{4}\, i\\ &= 4i(2i)\\ &= 8i^2\\ &= 8(-1)\\ &= -8 \quad \text{which is a false statement}\end{aligned}$$

Appendix A7 – Complex Numbers; Quadratic Equations in the Complex Number System (continued)

Quadratic equations with a negative discriminant have no real solution. But if we extend our number system to allow complex numbers, quadratic equations will always have a solution.

Theorem – In the complex number system, the solutions of the quadratic equation $ax^2 + bx + c = 0$,

where a, b, and c are real numbers and $a \neq 0$, are given by the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Example 10: Solve the equation $x^2 - 6x + 10 = 0$ in the complex number system.

$$
\begin{aligned}
&\Rightarrow\ a = 1,\ b = -6,\ c = 10\\
&x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\\
&x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(10)}}{2(1)}\\
&\ \ = \frac{6 \pm \sqrt{36 - 40}}{2}\\
&\ \ = \frac{6 \pm \sqrt{-4}}{2}\\
&\ \ = \frac{6 \pm \sqrt{4}\, i}{2}\\
&\ \ = \frac{6 \pm 2i}{2}\\
&\ \ = \frac{6}{2} \pm \frac{2i}{2}\\
&\ \ = 3 \pm i\\
&x = \{3 + i,\ 3 - i\}
\end{aligned}
$$

The discriminant gives you information about the types of solutions. In the complex number system, a quadratic equation $ax^2 + bx + c = 0$ with real coefficients has the following types of solutions:

1) If $b^2 - 4ac > 0$, the equation has two unequal real solutions.
2) If $b^2 - 4ac = 0$, the equation has one repeated real solution, or a double root.
3) If $b^2 - 4ac < 0$, the equation has two complex solutions that are not real.
   The solutions are conjugates of each other.

For Example 10 above, $b^2 - 4ac = (-6)^2 - 4(1)(10)$.

$$
\begin{aligned}
&= 36 - 40\\
&= -4\\
&< 0
\end{aligned}
$$

So you know that the solutions are complex numbers, and they are complex conjugates of each other.

All material has been taken from *Precalculus,* by M. Sullivan, 10th Edition