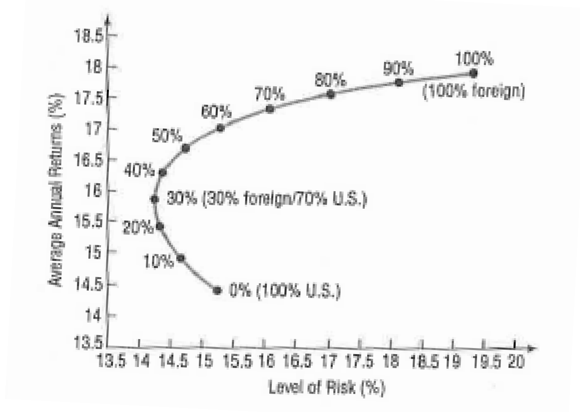
Section 1.2 – Graphs of Equations in Two Variables; Intercepts; Symmetry

Graph Equations by Plotting Points

An **equation in two variables**, say x and y, is a statement in which two expressions involving x and y are equal. The expressions are called the **sides** of the equation. Since an equation is a statement, it may be true or false, depending on the values of the variables. Any values of x and y that result in a true statement are said to **satisfy** the equation.

The **graph of an equation in two variables** x and y consists of the set of points in the xy-plane whose coordinates (x, y) satisfy the equation.

Graphs play an important role in helping us to visualize the relationships that exist between two variables or quantities. The figure below shows the relation between the level of risk in a stock portfolio and the average annual rate of return. From the graph, you can see that when 30% of a portfolio of stocks is invested in foreign companies, risk is minimized.



Example 1: Determine if the following points are on the graph of the equation $2x - y = 6$.

a) $(2, 3)$ b) $(2, -2)$

a) For the point $(2, 3)$, check to see whether $x = 2$ and $y = 3$ satisfies the equation.

The equation is / is not satisfied, so the point $(2, 3)$ is / is not on the graph of $2x - y = 6$.

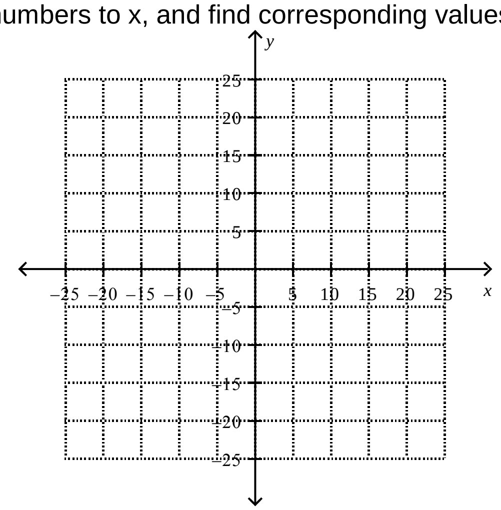
b) For the point $(2, -2)$, check to see whether $x = 2$ and $y = -2$ satisfies the equation.

The equation is / is not satisfied, so the point $(2, -2)$ is / is not on the graph of $2x - y = 6$.

Example 2: Graph the equation: $y = 2x + 5$

The graph consists of all points (x, y) that satisfy the equation. To locate some of these points (and get an idea of the pattern of the graph), assign some numbers to x, and find corresponding values for y.

x	y
−5	
0	
1	
10	



By plotting these points and then connecting them, you obtain the graph of the equation (a line).

Section 1.2 – Graphs of Equations in Two Variables; Intercepts; Symmetry (continued)

The graph in Example 2 does not show all points. For example, the point (20,45) is also a point on the line, but it is not shown. Since the graph of $y = 2x + 5$ can be extended out indefinitely, we use arrows to indicate that the pattern shown continues. It is important when illustrating a graph to present enough of the graph so that any viewer of the illustration will "see" the rest of it as an obvious continuation of what is actually there. This is referred to as a **complete graph**.

One way to obtain the complete graph of an equation is to plot enough points on the graph for a pattern to become evident. Then these points are connected with a smooth curve following the suggested pattern. But how many points are sufficient? Sometimes knowledge about the equation tells you.

One purpose of this text is to investigate the properties of equations in order to decide whether a graph is complete. Sometimes you will graph equations by plotting points. Another way to obtain the graph of an equation is to use a graphing utility. Two techniques that sometimes reduce the number of points required to graph an equation involve finding *intercepts* and checking for *symmetry*.

Find Intercepts from a Graph

The points, if any, at which a graph crosses or touches the coordinate axes are called the **intercepts**. The x-coordinate of a point at which the graph crosses or touches the x-axis is an **x-intercept**, and the y-coordinate of a point at which the graph crosses or touches the y-axis is a **y-intercept**.

Example 3: Find the intercepts of the graph in Figure 15. What are its x-intercepts? What are its y-intercepts?

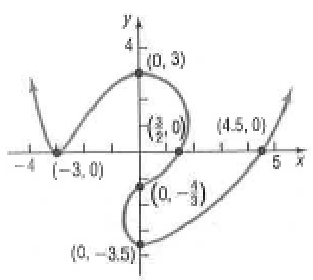


Figure 15

Solution: The intercepts of the graph are the points

The x-intercepts are

The y-intercepts are

If the type of intercept (x- versus y-) is not specified, then report the intercept as an ordered pair. However, if the type of intercept is specified, then you may report the coordinate of the specified intercept. For x-intercepts, report the x-coordinate of the intercept; for y-intercepts, report the y-coordinate of the intercept.

Find Intercepts from an Equation

The intercepts of a graph can be found from its equation by using the fact that points on the x-axis have y-coordinates equal to 0, and points on the y-axis have x-coordinates equal to 0.

Procedure for Finding Intercepts

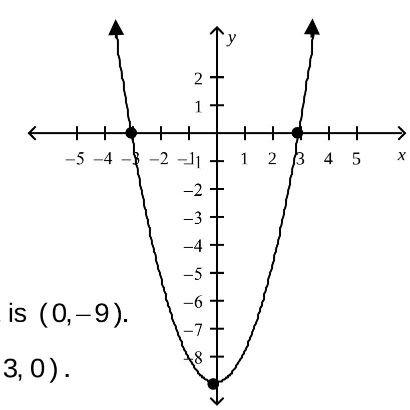
1) To find the x-intercept(s), if any, of the graph of an equation, set $y = 0$ in the equation and solve for x, where x is a real number.
2) To find the y-intercept(s), if any, of the graph of an equation, set $x = 0$ in the equation and solve for y, where y is a real number.

Example 4: Find the x- and y-intercepts of the function $y = x^2 - 9$.

x-intercept:
Set $y = 0$
$0 = x^2 - 9$
$x^2 = 9$
$x = \pm\sqrt{9}$
$x = \pm 3$

y-intercept:
Set $x = 0$
$y = 0^2 - 9$
$y = 0 - 9$
$y = -9$
The y-intercept is -9, so the point is $(0, -9)$.

The x-intercepts are -3 and 3, so the points are $(-3, 0)$ and $(3, 0)$.

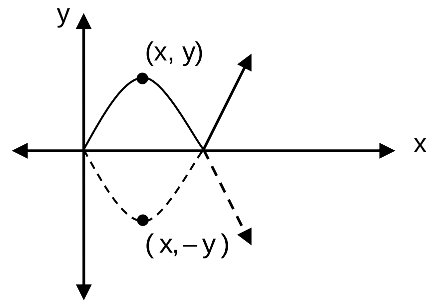


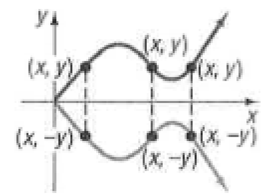
Section 1.2 – Graphs of Equations in Two Variables; Intercepts; Symmetry (continued)

Test an Equation for Symmetry with Respect to the x-Axis, the y-Axis, and the Origin

Another tool for graphing equations involves symmetry, particularly symmetry with respect to the x-axis, the y-axis, and the origin. Symmetry often occurs in nature, such as a butterfly.

A graph is said to be symmetric with respect to the x-axis if, for every point (x, y) on the graph, the point $(x, -y)$ is also on the graph.

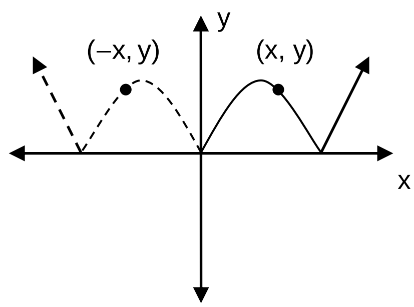


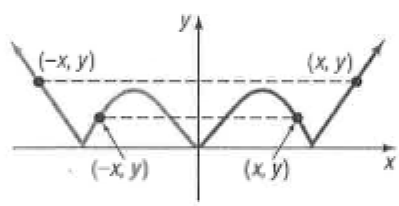


When a graph is symmetric with respect to the x-axis, the part of the graph above the x-axis is a reflection or mirror image of the part below it, and vice versa.

Example 5: If a graph is symmetric with respect to the x-axis, and the point (3,2) is on the graph, then the point $(3, -2)$ is also on the graph.

A graph is said to be symmetric with respect to the y-axis if, for every point (x, y) on the graph, the point $(-x, y)$ is also on the graph.

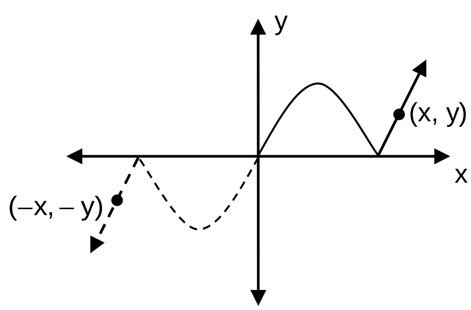


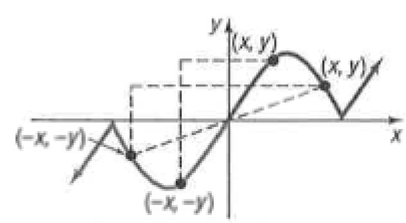


When a graph is symmetric with respect to the y-axis, the part of the graph to the right of the y-axis is a reflection or mirror image of the part to the left of it, and vice versa.

Example 6: If a graph is symmetric with respect to the y-axis, and the point (5,8) is on the graph, then the point $(-5, 8)$ is also on the graph.

A graph is said to be symmetric with respect to the origin if, for every point (x, y) on the graph, the point $(-x, -y)$ is also on the graph.





When a graph is symmetric with respect to the origin, it may be viewed in three ways:

1) As a reflection about the y-axis, followed by a reflection about the x-axis.
2) As a projection along a line through the origin so that the distances from the origin are equal.
3) As half of a complete revolution about the origin.

Example 7: If a graph is symmetric with respect to the origin, and the point (4,2) is on the graph, then the point $(-4, -2)$ is also on the graph.

Section 1.2 – Graphs of Equations in Two Variables; Intercepts; Symmetry (continued)

When the graph of an equation is symmetric with respect to a coordinate axis or the origin, the number of points that you need to plot in order to see the pattern is reduced. For example, if the graph of an equation is symmetric with respect to the y-axis, then once points to the right of the y-axis are plotted, an equal number of points on the graph can be obtained by reflecting them about the y-axis. Because of this, before you graph an equation, you first want to determine whether it has any symmetry. The following tests are used for this purpose.

To test the graph of an equation for symmetry with respect to the :

x-axis: Replace y with $-y$ in the equation and simplify. If an equivalent equation results, then the graph of the equation is symmetric with respect to (wrt) the x-axis.

y-axis: Replace x with $-x$ in the equation and simplify. If an equivalent equation results, then the graph of the equation is symmetric with respect to (wrt) the y-axis.

origin: Replace x with $-x$ and y with $-y$ in the equation and simplify. If an equivalent equation results, then the graph of the equation is symmetric with respect to (wrt) the origin.

Example 8: Graph $y = 2x^3$. Find any intercepts, and check for symmetry.

x-intercept: Set $y = 0$
$0 = 2x^3$
=> $x = 0$

y-intercept: Set $x = 0$
$y = 2(0)^3$
=> $y = 0$

So, the only intercept is (0, 0).

Symmetry: x-axis: Replace y with $-y$. $-y = 2x^3$
=> $y = -2x^3$

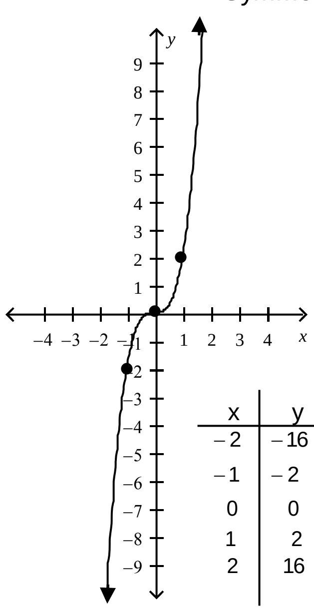
This is not equivalent to the original $y = 2x^3$ => not symmetric wrt the x-axis

y-axis: Replace x with $-x$. $y = 2(-x)^3$
=> $y = -2x^3$

This is not equivalent to the original $y = 2x^3$ => not symmetric wrt the y-axis

origin: Replace x with $-x$ and y with $-y$. $-y = 2(-x)^3$
=> $-y = -2x^3$
$y = 2x^3$

This is equivalent to the original $y = 2x^3$ => symmetric wrt the origin
Thus, (x, y) corresponds to $(-x, -y)$.



x	y
−2	−16
−1	−2
0	0
1	2
2	16

Section 1.2 – Graphs of Equations in Two Variables; Intercepts; Symmetry (continued)

Know How to Graph Key Equations

The next three examples use intercepts, symmetry, and point plotting to obtain the graphs of key equations. It is important to know the graphs of these key equations because we will use them later.

Example 9: Graph the equation $x = y^3$.

Graph the equation $y = x^3$ by plotting points. Find any intercepts and check for symmetry first.

Solution First, find the intercepts. When $x = 0$, then $y = 0$; and when $y = 0$, then $x = 0$. The origin $(0, 0)$ is the only intercept. Now test for symmetry.

x-Axis: Replace y by $-y$. Since $-y = x^3$ is not equivalent to $y = x^3$, the graph is not symmetric with respect to the x-axis.

y-Axis: Replace x by $-x$. Since $y = (-x)^3 = -x^3$ is not equivalent to $y = x^3$, the graph is not symmetric with respect to the y-axis.

Origin: Replace x by $-x$ and y by $-y$. Since $-y = (-x)^3 = -x^3$ is equivalent to $y = x^3$ (multiply both sides by -1), the graph is symmetric with respect to the origin.

To graph $y = x^3$, use the equation to obtain several points on the graph. Because of the symmetry, we need to locate only points on the graph for which $x \geq 0$. See Table 3. Since $(1, 1)$ is on the graph, and the graph is symmetric with respect to the origin, the point $(-1, -1)$ is also on the graph. Plot the points from Table 3 and use the symmetry. Figure 21 shows the graph.

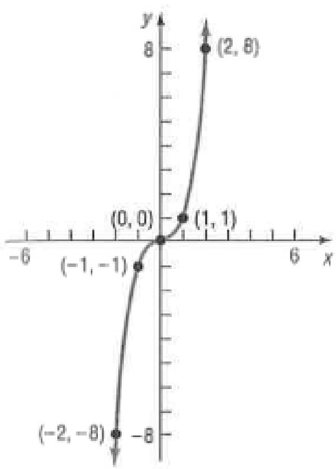


Figure 21 $y = x^3$

Table 3

x	$y = x^3$	(x, y)
0	0	(0, 0)
1	1	(1, 1)
2	8	(2, 8)
3	27	(3, 27)

Example 10: Graph the equation $x = y^2$.

(a) Graph the equation $x = y^2$. Find any intercepts and check for symmetry first.
(b) Graph $x = y^2$, $y \geq 0$.

Solution (a) The lone intercept is $(0, 0)$. The graph is symmetric with respect to the x-axis. (Do you see why? Replace y by $-y$.) Figure 22 shows the graph.

(b) If we restrict y so that $y \geq 0$, the equation $x = y^2$, $y \geq 0$, may be written equivalently as $y = \sqrt{x}$. The portion of the graph of $x = y^2$ in quadrant I is therefore the graph of $y = \sqrt{x}$. See Figure 23.

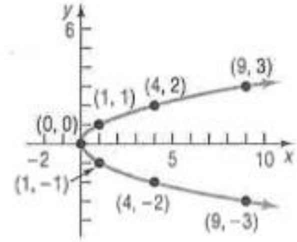


Figure 22 $x = y^2$

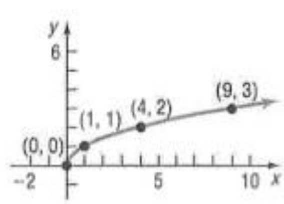


Figure 23 $y = \sqrt{x}$

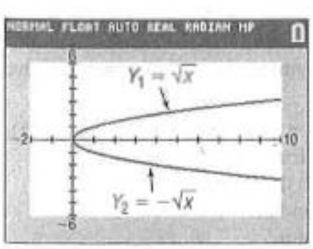


Figure 24

COMMENT To see the graph of the equation $x = y^2$ on a graphing calculator, you will need to graph two equations: $Y_1 = \sqrt{x}$ and $Y_2 = -\sqrt{x}$. We discuss why in Chapter 2. See Figure 24.

Section 1.2 – Graphs of Equations in Two Variables; Intercepts; Symmetry (continued)

Example 11: Graph the equation $y = \frac{1}{x}$.

Graph the equation $y = \frac{1}{x}$. First, find any intercepts and check for symmetry.

Solution Check for intercepts first. If we let $x = 0$, we obtain 0 in the denominator, which makes y undefined. We conclude that there is no y-intercept. If we let $y = 0$, we get the equation $\frac{1}{x} = 0$, which has no solution. We conclude that there is no x-intercept. The graph of $y = \frac{1}{x}$ does not cross or touch the coordinate axes.

Next check for symmetry:

x-Axis: Replacing y by $-y$ yields $-y = \frac{1}{x}$, which is not equivalent to $y = \frac{1}{x}$.

y-Axis: Replacing x by $-x$ yields $y = \frac{1}{-x} = -\frac{1}{x}$, which is not equivalent to $y = \frac{1}{x}$.

Origin: Replacing x by $-x$ and y by $-y$ yields $-y = -\frac{1}{x}$, which is equivalent to $y = \frac{1}{x}$. The graph is symmetric with respect to the origin.

Now set up Table 4, listing several points on the graph. Because of the symmetry with respect to the origin, we use only positive values of x. From Table 4 we infer that if x is a large and positive number, then $y = \frac{1}{x}$ is a positive number close to 0. We also infer that if x is a positive number close to 0, then $y = \frac{1}{x}$ is a large and positive number. Armed with this information, we can graph the equation.

Figure 25 illustrates some of these points and the graph of $y = \frac{1}{x}$. Observe how the absence of intercepts and the existence of symmetry with respect to the origin were utilized.

Table 4

x	$y = \frac{1}{x}$	(x, y)
$\frac{1}{10}$	10	$\left(\frac{1}{10}, 10\right)$
$\frac{1}{3}$	3	$\left(\frac{1}{3}, 3\right)$
$\frac{1}{2}$	2	$\left(\frac{1}{2}, 2\right)$
1	1	$(1, 1)$
2	$\frac{1}{2}$	$\left(2, \frac{1}{2}\right)$
3	$\frac{1}{3}$	$\left(3, \frac{1}{3}\right)$
10	$\frac{1}{10}$	$\left(10, \frac{1}{10}\right)$

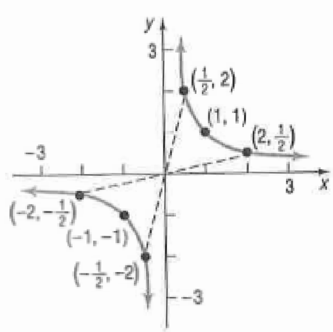


Figure 25 $y = \frac{1}{x}$

COMMENT Refer to Example 2 in Appendix B, Section B.3, for the graph of $y = \frac{1}{x}$ found using a graphing utility. ■

All material has been taken from *Precalculus,* by M. Sullivan, 10th Edition