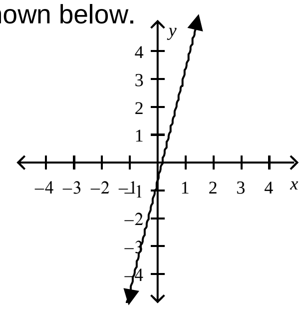
Section 2.1 – Functions – Day 1

Often there are situations where the value of one variable is somehow linked to the value of another variable. For example, a car's engine size is linked to its gas mileage. When the value of one variable is related to the value of a second variable, you have a relation.

A relation is defined to be a correspondence between two sets. If x and y are two elements, one from each of these sets, and if a relation exists between x and y, then we say that x **corresponds** to y or that y **depends on** x, and we write $x \rightarrow y$.

There are a number of ways to express relations between two sets. For example, the equation $y = 4x - 1$ shows a relation between x and y. It says that if you take some number *x*, multiply it by 4, and then subtract 1, you obtain the corresponding value of *y*. In this sense, *x* serves as the **input** to the relation, and *y* is the **output** of the relation. This relation, expressed as a graph, is shown below.



The set of all inputs for a relation is called the **domain** of the relation, and the set of all outputs is called the **range**.

In addition to being expressed as equations and graphs, relations can be expressed through a technique called *mapping*. A **map** illustrates a relation as a set of inputs with an arrow drawn from each element in the set of inputs to the corresponding element in the set of outputs. **Ordered pairs** can be used to represent $x \rightarrow y$ as (x, y). When relations are written as ordered pairs (x, y), we say that x is related to y. We are interested in the type of relation that might exist between the two variables.

Example 1: Figure 2 shows a relation between states and the number of representatives each state has in the House of Representatives. The relation might be named "number of representatives."

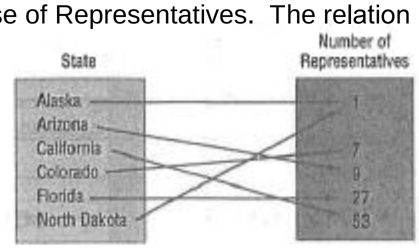


Figure 2 Number of representatives

In this relation, Alaska corresponds to 1, Arizona corresponds to 9, and so on. Using ordered pairs, this relation would be expressed as

{ (Alaska, 1), (Arizona, 9), (California, 53), (Colorado, 7), (Florida, 27), (North Dakota, 1) }

The domain of the relation is { Alaska, Arizona, California, Colorado, Florida, North Dakota }, and the range is { 1, 7, 9, 27, 53 }.

One of the most important concepts in algebra in the *function*. A function is a special type of relation. To understand the idea behind a function, look again at Example 1. If you were asked "how many representatives does Alaska have?" your answer would be one. Each input *state* corresponds to a single output *number of representatives*.

Consider a second relation shown in Figure 3. This relation involves a correspondence between four people and their phone numbers. Notice that Colleen has two phone numbers, so there is no single answer to the question "what is Colleen's phone number?"

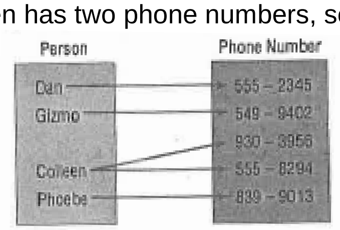


Figure 3 Phone numbers

Section 2.1 – Functions – Day 1 (Continued)

Let's look at one more relation. Figure 4 is a relation that shows a correspondence between type of *animal* and *life expectancy*. If asked to determine the life expectancy of a dog, you would say 11 years. If asked to determine the life expectancy of a rabbit, you would say 7 years.

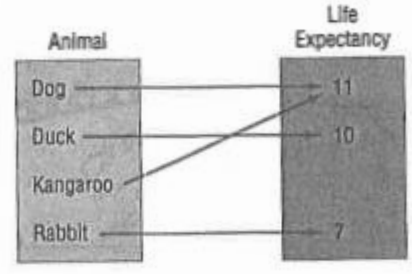


Figure 4 Animal life expectancy

Notice that for the relations in both Figures 2 and 4, each input corresponds to exactly one output. This leads to the definition of a *function*.

Definition of a Function: Let X and Y be two nonempty sets of real numbers. A **function** from X into Y is a relation that associates with each element of X exactly one element of Y.

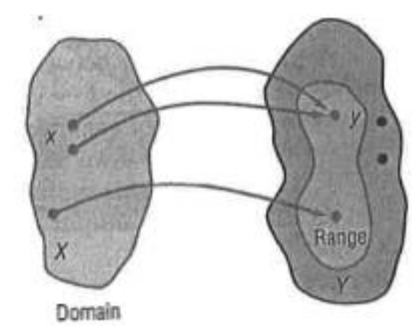


Figure 5

The set X is called the **domain** of the function. For each element *x* in X, the corresponding element *y* in Y is called the value of the function at **x**, or the image of *x*. The set of all images of the elements in the domain is called the **range** of the function. See Figure 5.

Since there may be some elements in Y that are not the image of some x in X, it follows that the range of a function may be a subset of Y, as shown in Figure 5.

Not all relations between two sets are functions. The next example shows how to determine whether a relation is a function.

Example 2:

For each relation in Figures 6, 7, and 8, state the domain and range. Then determine whether the relation is a function.

Figure 6: The input is the number of calories in a fast-food sandwich, and the output is the fat content (in grams).

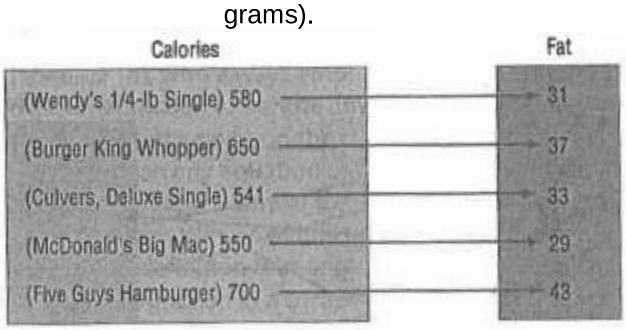


Figure 6 Fat content
Source: Each company's Website

The domain of the relation is { 541, 550, 580, 650, 700 }, and the range of the relation is { 29, 31, 33, 37, 43 }.

This relation is a function because each element in the domain corresponds to exactly one element in the range.

Section 2.1 – Functions – Day 1 (Continued)

Figure 7: The inputs are the gasoline stations in Harris County, Texas, and the outputs are the price per gallon of unleaded regular in March 2014.

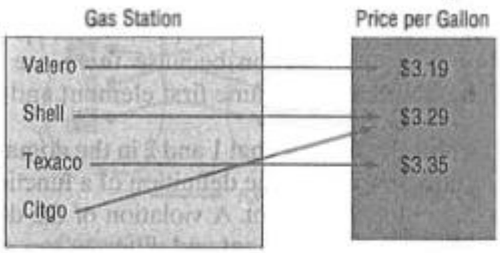


Figure 7 Unleaded price per gallon

The domain of the relation is { Citgo, Shell, Texaco, Valero }.

The range is { \$3.19, \$3.29, \$3.35 }.

This relation is a function because each element in the domain corresponds to exactly one element in the range.

Notice that it is okay for more than one element in the domain to correspond to the same element in the range (Shell and Citgo both sell gas for \$3.29 a gallon).

Figure 8: The inputs are the weight (in carats) of pear-cut diamonds and the outputs are the price (in dollars).

Figure 8 Diamond price
Source: Used with permission of Diamonds.com

The domain of the relation is { 0.70, 0.71, 0.75, 0.78 }.

The range is { \$1529, \$1575, \$1765, \$1798, \$1952 }.

This relation is a not a function because not every element in the domain corresponds to exactly one element in the range.
If a 0.71-carat diamond is chosen from the domain, a single price cannot be assigned to it.

So, for a function, no input has more than one output. The domain of a function is the set of all inputs; the range is the set of all outputs.

You may also think of a function as a set of ordered pairs (x, y) in which no ordered pairs have the same first element and different second elements. The set of all first elements *x* is the domain of the function, and the set of all second elements *y* is its range. Each element *x* in the domain corresponds to exactly one element *y* in the range.

Example 3: Determine whether each relation represents a function. State the domain and range of each.

A) $\{(1,3),\ (2,-7),\ (3,10),\ (4,3)\}$

Domain: {1, 2, 3, 4 } Range: { –7 , 3, 10 }

Yes, this relation is a function since each input x has a unique output y. There are no ordered pairs with the same first element and different second elements.

B) $\{(3,7),\ (2,-4),\ (3,-5),\ (6,2)\}$

Domain: { 2, 3, 6 } Range: { –5, –4, 2, 7 }

No, this relation is not a function since each input x does not have a unique output y. The ordered pairs $(3,7)$ and $(3,-5)$ have the same first element and different second elements.

C) $\{(1,3),\ (2,5),\ (6,5)\}$

Domain: {1, 2, 6 } Range: { 3, 5 }

Yes, this relation is a function since each input x has a unique output y. There are no ordered pairs with the same first element and different second elements. Notice that 2 and 6 in the domain both have the same image in the range. This does not violate the definition of a function; two different first elements can have the same second element. A violation occurs when two ordered pairs have the same first element and different second elements, as is B) above.

Section 2.1 – Functions – Day 1 (Continued)

Example 4: Determine whether the equation $y = 3x + 7$ is a function.

The equation tells you to take an input x, multiply it by 3, and then add 7. For any input x, these operations yield only one output y, so the equation is a function.

For example, if $x = 1$, then

$$y = 3(1) + 7$$
$$y = 10$$

if $x = 4$, then

$$y = 3(4) + 7$$
$$y = 12 + 7$$
$$y = 19$$

The graph of the equation is a line with slope 3 and y-intercept 7.
The function is called a *linear function.*

Example 5: Determine whether the equation $x^2 + y^2 = 4$ defines *y* as a function of *x*.

To determine whether the equation $x^2 + y^2 = 4$, which is a circle centered at the origin with radius 2, is a function, solve the equation for *y*.

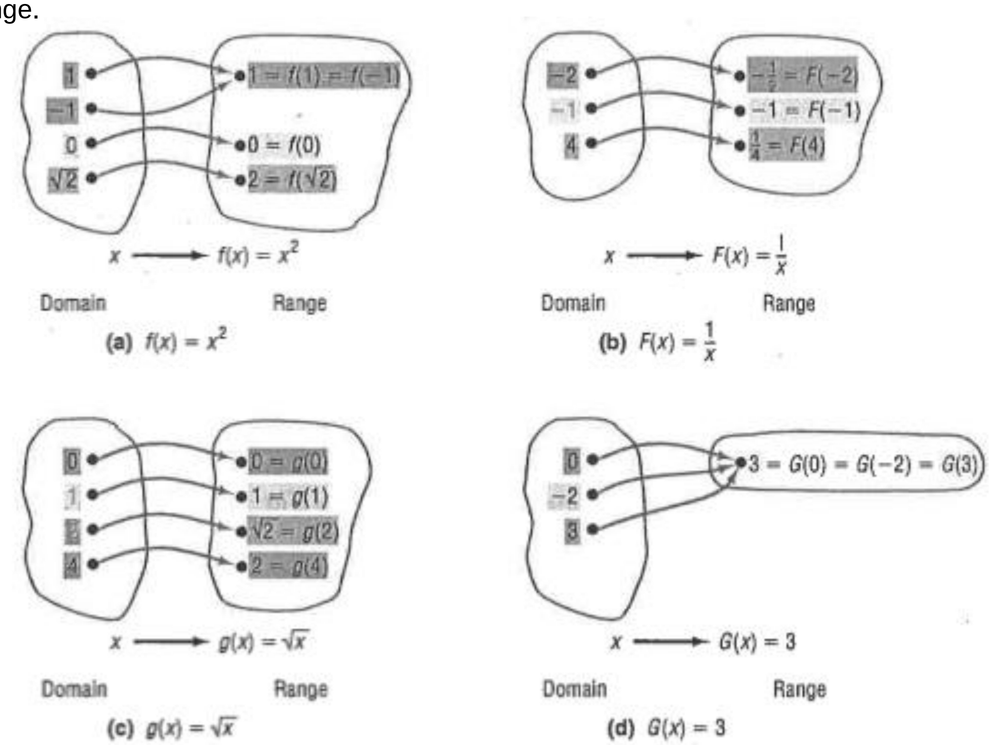
$$x^2 + y^2 = 4$$
$$y^2 = 4 - x^2$$
$$y = \pm\sqrt{4 - x^2}$$

For each x-value between -2 and 2 ($-2 \le x \le 2$), there are two values of y. Thus, the equation $x^2 + y^2 = 4$ does not define a function.

Find the Value of a Function

Functions are often denoted by letters such as *f*, F, *g*, G and others. If f is a function, then for each number x in its domain, the corresponding image in the range is designated by the symbol $f(x)$, read as "f of x" or as "f at x." We refer to $f(x)$ as the **value of f at the number x.** So, $f(x)$ is the number that results when x is given and the function f is evaluated at x. $f(x)$ is the output corresponding to x or $f(x)$ is the image of x. $f(x)$ does not mean "f times x." For example, the function in Example 4 may be written as $f(x) = 3x + 7$. Then $f(1) = 10$ and $f(4) = 19$.

Figure 9 illustrates some other functions. Notice that in every function, for each x in the domain, there is one value in the range.



(a) $f(x) = x^2$ (b) $F(x) = \frac{1}{x}$ (c) $g(x) = \sqrt{x}$ (d) $G(x) = 3$

Figure 9

Section 2.1 – Functions – Day 1 (Continued)

Sometimes it is helpful to think of a function f as a machine that receives as input a number from the domain, manipulates it, and outputs a value. See Figure 10.

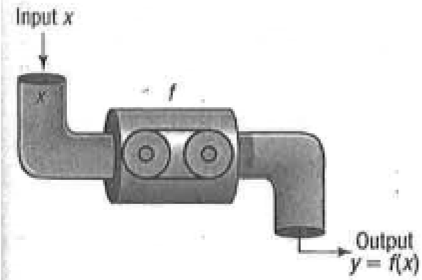


Figure 10 Input/output machine

The restrictions on this input/output machine are as follows:

1) It accepts only numbers from the domain of the function.
2) For each input, there is exactly one output (which may be repeated for different inputs).

For a function $y = f(x)$, the variable x is called the **independent variable**, because it can be assigned any of the permissible numbers from the domain. The independent variable x is also called the argument of the function. The variable y is called the **dependent variable**, because its value depends on x. The dependent variable y is said to be the value of f at x.

Any symbols can be used to represent the independent and dependent variables. For example, if f is the *cube function*, then f can be given by $f(x) = x^3$ or $f(t) = t^3$ or $f(z) = z^3$. All three functions are the same. Each says to cube the independent variable to get the output.

The independent variable is also called the **argument** of the function. Thinking of the independent variable as an argument can sometimes make it easier to find the value of a function. For example, if f is the function defined by $f(x) = x^3$, then f tells you to cube the argument. Thus $f(2)$ means to cube 2, $f(A)$ means to cube the number A, and $f(x+h)$ means to cube the quantity $x+h$.

Example 6: Evaluate each of the following for the function $F(x) = x^2 + 4x$.

a) $F(2)$

$$F(2) = 2^2 + 4(2) = 4 + 8 = 12$$

b) $F(x) + F(2)$

$$F(x) + F(2) = x^2 + 4x + (2)^2 + 4(2) = x^2 + 4x + 4 + 8 = x^2 + 4x + 12$$

c) $F(x+2)$

$$F(x+2) = (x+2)^2 + 4(x+2) = x^2 + 4x + 4 + 4x + 8 = x^2 + 8x + 12$$

Notice that $F(x+2) \neq F(x) + F(2)$

d) $-F(x)$

$$-F(x) = -\left(x^2 + 4x\right) = -x^2 - 4x$$

e) $F(-x)$

$$F(-x) = (-x)^2 + 4(-x) = x^2 - 4x$$

Notice that $-F(x) \neq F(-x)$

f) $2F(x)$

$$2F(x) = 2\left(x^2 + 4x\right) = 2(x^2) + 2(4x) = 2x^2 + 8x$$

g) $F(2x)$

$$F(2x) = (2x)^2 + 4(2x) = 4x^2 + 8x$$

Notice that $2F(x) \neq F(2x)$

Most calculators have special keys that allow you to find the value of certain commonly used functions. For example, you should be able to find the square function $f(x) = x^2$, the square root function $f(x) = \sqrt{x}$, the reciprocal function $f(x) = \frac{1}{x} = x^{-1}$, and others that will be used later in the text.

Section 2.1 – Functions – Day 1 (Continued)

In general, when a function f is defined by an equation in x and y, we say that the function f is given **implicitly**. If it is possible to solve the equation for y in terms of x, we write $y = f(x)$ and say that the function is given **explicitly**.

Example 7:

Implicit Form	**Explicit Form**
$4x + y = 7$	$y = -4x + 7$ or $f(x) = -4x + 7$
$x^2 - y = 12$	$y = x^2 - 12$ or $f(x) = x^2 - 12$
$2x^2 y = 9$	$y = \dfrac{9}{2x^2}$ or $f(x) = \dfrac{9}{2x^2}$

Summary

Important Facts about Functions

1) For each *x* in the domain of a function f, there is exactly one image $f(x)$ in the range; however, an element in the range can result from more than one *x* in the domain.

2) f is the symbol that we use to denote the function. It is symbolic of the equation (rule) that we use to get from an *x* in the domain to $f(x)$ in the range.

3) If $y = f(x)$ then *x* is called the independent variable or argument of f, and *y* is called the dependent variable or the value of f at x.

All material has been taken from *Precalculus,* by M. Sullivan, 10th Edition