Section 2.1 – Functions – Day 2

Find the Difference Quotient of a Function

An important concept in calculus involves looking at a certain quotient. For a given function $y = f(x)$, the inputs x and $x+h$, $h \neq 0$, result in the images $f(x)$ and $f(x+h)$. The quotient of their differences $\frac{f(x+h)-f(x)}{(x+h)-x} = \frac{f(x+h)-f(x)}{h}$ with $h \neq 0$, is called the **difference quotient of f** at x.

The **difference quotient** of a function f at *x* is given by $\frac{f(x+h)-f(x)}{h}$ $\quad h \neq 0$.

The difference quotient is used in calculus to define the derivative, which leads to applications such as the velocity of an object and optimization resources.

When finding a difference quotient, it is necessary to simplify the expression in order to cancel the h in the denominator, as shown in the following example.

Example 7: Find the difference quotient of each function.

a) $f(x) = 3x^2 - 4x$

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{\left(3(x+h)^2 - 4(x+h)\right) - \left(3x^2 - 4x\right)}{h} \\
&= \frac{3(x^2 + 2xh + h^2) - 4x - 4h - 3x^2 + 4x}{h} \\
&= \frac{\cancel{3x^2} + 6xh + 3h^2 - \cancel{4x} - 4h - \cancel{3x^2} + \cancel{4x}}{h} \\
&= \frac{3h^2 + 6xh - 4h}{h} \\
&= \frac{(3h + 6x - 4)\cancel{h}}{\cancel{h}} \\
&= 6x + 3h - 4
\end{aligned}$$

b) $f(x) = \frac{3}{x}$

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{\frac{3}{x+h} - \frac{3}{x}}{h} \\
&= \frac{\frac{3x - 3(x+h)}{x(x+h)}}{h} \\
&= \frac{\frac{3x - 3x - 3h}{x(x+h)}}{h} \\
&= \frac{\frac{-3h}{x(x+h)}}{h} \\
&= \frac{-3h}{xh(x+h)} \\
&= \frac{-3}{x(x+h)}
\end{aligned}$$

c) $f(x) = \sqrt{x}$

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \frac{\sqrt{x+h} - \sqrt{x}}{h}\left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right) \\
&= \frac{\left(\sqrt{x+h}\right)^2 - \left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h} + \sqrt{x}\right)} \\
&= \frac{x+h-x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} \\
&= \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)} \\
&= \frac{1}{\sqrt{x+h} + \sqrt{x}}
\end{aligned}$$

Section 2.1 – Functions – Day 2 (continued)

Find the Domain of a Function Defined by an Equation

Often the domain of a function f is not specified; instead, only the equation defining the function is given.

Domain of a function f: If not specified, the domain of f is the largest set of real numbers for which the value $f(x)$ is a real number.

Example 8: Find the domain of each function.

a) $f(x)=x^2-2x$

The domain of f is the set of all real numbers.

b) $g(x)=\dfrac{2}{x^2-9}$

Division by zero is not allowed, so the denominator x^2-9 cannot be zero. Thus, $x\neq\pm3$.

The domain of g $=\{x \mid x\neq-3 \text{ and } x\neq3\}$.

c) $h(x)=\sqrt{5x+3}$

Since only nonnegative numbers have real square roots, $5x+3\geq0$.

$$5x\geq-3$$

$$x\geq\frac{-3}{5}$$

The domain of h $=\left\{x \mid x\geq\dfrac{-3}{5}\right\}$ or $\left[\dfrac{-3}{5},\ \infty\right)$.

d) $k(x)=\dfrac{\sqrt{2x-8}}{x+6}$

First, since only nonnegative numbers have real square roots, $2x-8\geq0$.

$$2x\geq8$$

$$x\geq4$$

Also, since division by zero is undefined, $x+6\neq0$. So, $x\neq-6$.

The domain of k $=\{x \mid x\geq4\}$ or $[4,\ \infty)$.

You may find the following steps helpful for finding the domain of a function that is defined by an equation and whose domain is a subset of the real numbers.

Finding the Domain of a Function Defined by an Equation

1) Start with the domain as the set of real numbers.
2) If the equation has a denominator, exclude any numbers that give a zero denominator.
3) If the equation has a radical of even index, exclude any numbers that cause the expression inside the radical (the radicand) to be negative.

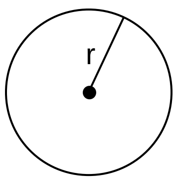
If x is in the domain of a function f, we say that **f is defined at x**, or **$f(x)$ exists**. If x is not in the domain of f, we say that **f is not defined at x**, or **$f(x)$ does not exist**. For example, if $g(x)=\dfrac{2x}{x^2-9}$, $g(0)$ exists, but $g(-3)$ and $g(3)$ do not exist.

Section 2.1 – Functions – Day 2 (continued)

We have not said much about finding the range of a function. We will say more about finding the range when we look at the graph of a function in the next section. When a function is defined by an equation, it can be difficult to find the range. Therefore, we shall usually be content to find just the domain of a function when the function is defined by an equation. We will express the domain of a function using inequalities, interval notation, set notation, or words, whichever is most convenient.

When you use functions in applications, the domain may be restricted by physical or geometric considerations. For example, if x is the domain variable and it represents the length of a figure's side, then the domain of the function is restricted to the positive real numbers, since the length of a side must be greater than 0. So, always check your answers to be certain the answer is reasonable or makes sense.

Example 9: Express the area of a circle as a function of its radius. Find the domain.



The figure shows a circle of radius r. The formula for the area of a circle of radius r is $A = \pi r^2$. Using r to represent the independent variable and A to represent the dependent variable, the function expressing this relationship is $A(r) = \pi r^2$. Here, the domain is $\{r | r > 0\}$.

Form the Sum, Difference, Product, and Quotient of Two Functions

Now, we introduce some operations on functions. Functions, like numbers, can be added, subtracted, multiplied, and divided.

The symbol $\cap$ stands for intersection. It means you should find the elements that are common to two sets.

If f and g are functions:

Their sum, $f + g$, is the function defined by $(f+g)(x) = f(x) + g(x)$.
The domain of $(f+g)(x)$ consists of the numbers x that are in the domain of f and in the domain of g.
That is, the domain consists of the numbers x that are in the domains of both f and g. So,
domain of $f+g$ = domain of f $\cap$ domain of g.

Their difference, $f - g$, is the function defined by $(f-g)(x) = f(x) - g(x)$.
The domain of $(f-g)(x)$ consists of the numbers x that are in the domain of f and in the domain of g.
That is, the domain consists of the numbers x that are in the domains of both f and g. So,
domain of $f-g$ = domain of f $\cap$ domain of g.

Their product $f \bullet g$, is the function defined by $(f \bullet g)(x) = f(x) \bullet g(x)$.
The domain of $(f \bullet g)(x)$ consists of the numbers x that are in the domain of f and in the domain of g.
That is, the domain consists of the numbers x that are in the domains of both f and g. So,
domain of $f \bullet g$ = domain of f $\cap$ domain of g.

Their quotient, $\frac{f}{g}$, is the function defined by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, where $g(x) \neq 0$.

The domain of $\left(\frac{f}{g}\right)(x)$ consists of the numbers x for which $g(x) \neq 0$ that are in the domain of f and in the domain of g. That is, the domain consists of the numbers x that are in the domains of both f and g.

So, domain of $\left(\frac{f}{g}\right)(x)$ = $\{x | g(x) \neq 0\} \cap$ domain of f $\cap$ domain of g.

Section 2.1 – Functions – Day 2 (continued)

Example 9: Let f and g be two functions defined as $f(x)=5x^2$ and $g(x)=3x-1$.
Find all four operations of f and g and the domain for each operation.

a) $(f+g)(x)=f(x)+g(x)$
$$=5x^2+3x-1$$
Domain: All real numbers

b) $(f-g)(x)=f(x)-g(x)$
$$=5x^2-(3x-1)$$
$$=5x^2-3x+1$$
Domain: All real numbers

c) $(f\bullet g)(x)=f(x)\bullet g(x)$
$$=5x^2(3x-1)$$
$$=15x^3-5x^2$$
Domain: All real numbers

d) $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$
$$=\frac{5x^2}{3x-1}$$
Domain: All real numbers except $x=\frac{1}{3}$

OR $\left\{x \,\middle|\, x\neq\frac{1}{3}\right\}$

Example 10: Let f and g be two functions defined as $f(x)=\frac{2}{x-2}$ and $g(x)=\frac{x}{x+1}$.
Find all four operations of f and g and the domain for each operation.

a) $(f+g)(x)=f(x)+g(x)$
$$=\frac{2}{x-2}+\frac{x}{x+1}$$
$$=\frac{2}{x-2}\left(\frac{x+1}{x+1}\right)+\frac{x}{x+1}\left(\frac{x-2}{x-2}\right)$$
$$=\frac{2x+2}{(x-2)(x+1)}+\frac{x^2-2x}{(x+1)(x-2)}$$
$$=\frac{x^2+2}{(x+1)(x-2)}$$
Domain: $\{\,x \mid x\neq-1 \text{ and } x\neq 2\,\}$

b) $(f-g)(x)=f(x)-g(x)$
$$=\frac{2}{x-2}-\frac{x}{x+1}$$
$$=\frac{2}{x-2}\left(\frac{x+1}{x+1}\right)-\frac{x}{x+1}\left(\frac{x-2}{x-2}\right)$$
$$=\frac{2x+2}{(x-2)(x+1)}-\frac{x^2-2x}{(x+1)(x-2)}$$
$$=\frac{-x^2+4x+2}{(x+1)(x-2)}$$
Domain: $\{\,x \mid x\neq-1 \text{ and } x\neq 2\,\}$

c) $(f\bullet g)(x)=f(x)\bullet g(x)$
$$=\frac{2}{x-2}\left(\frac{x}{x+1}\right)$$
$$=\frac{2x}{(x-2)(x+1)}$$
Domain: $\{\,x \mid x\neq-1 \text{ and } x\neq 2\,\}$

d) $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$
$$=\frac{\frac{2}{x-2}}{\frac{x}{x+1}}$$
$$=\frac{2}{x-2}\left(\frac{x+1}{x}\right)$$
$$=\frac{2x+2}{x(x-2)}$$
Domain: $\{\,x \mid x\neq 0 \text{ and } x\neq 2\,\}$

In calculus, it is sometimes helpful to view a complicated function as the sum, difference, product, or quotient of simpler functions. For example, $F(x)=x^2+\sqrt{x}$ is the sum of $f(x)=x^2$ and $g(x)=\sqrt{x}$

Summary

Function: A relation between two sets of real numbers so that each number *x* in the first set, the domain, has corresponding to it exactly one number *y* in the second set, the range.

A set of ordered pairs (x, y) or $(x, F(x))$ in which no first element is paired with two different second elements.

The range is the set of y-values of the function that are the images of the x-values in the domain.

A function f may be defined implicitly by an equation involving *x* and *y* or explicitly by writing $y = f(x)$.

Unspecified Domain: If a function f is defined by an equation and no domain is specified, then the domain will be taken to be the largest set of real numbers for which the equation defines a real number.

Function Notation: $y = f(x)$

f is a symbol for the function.

x is the independent variable, or argument.

y is the dependent variable.

$f(x)$ is the value of the function at x, or the image of x.

All material has been taken from *Precalculus,* by M. Sullivan, 10th Edition