Section 2.2 – The Graph of a Function

Graphs often demonstrate more clearly the relationship between two variables than an equation or table, because graphs give a "visual picture." Equations and tables usually require some calculations and interpretation before information can be "seen." For Example, Table 1 shows the average price of gasoline in the United States for the years 1985-2014 (adjusted for inflation, based on 2014 dollars). If you plot these data and then connect the points, you obtain Figure 13.

Table 1

Year	Price	Year	Price	Year	Price
1985	2.55	1995	1.72	2005	2.74
1986	1.90	1996	1.80	2006	3.01
1987	1.89	1997	1.76	2007	3.19
1988	1.81	1998	1.49	2008	3.56
1989	1.87	1999	1.61	2009	2.58
1990	2.03	2000	2.03	2010	3.00
1991	1.91	2001	1.90	2011	3.69
1992	1.82	2002	1.76	2012	3.72
1993	1.74	2003	1.99	2013	3.54
1994	1.71	2004	2.31	2014	3.43

Source: U.S. Energy Information Administration

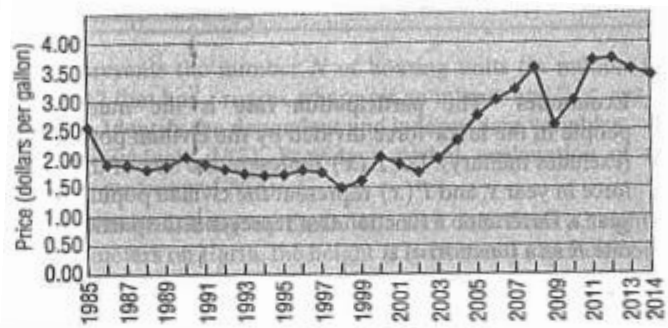


Figure 13 Average retail price of gasoline (2014 dollars)
Source: U.S. Energy Information Administration

You can see from the graph that the price of gasoline stayed roughly the same from 1986 to 1991 and rose rapidly from 2002 to 2008. The graph also shows that the lowest price occurred in 1998. To learn information such as this from an equation requires that some calculations be made.

Look again at Figure 13. The graph shows that for each date on the horizontal axis, there is only one price on the vertical axis. The graph represents a function, although the exact rule for getting from date to price is not given.

When a function is defined by an equation in x and y, the graph of the function is the graph of the equation; that is, it is the set of points (x, y) in the xy-plane that satisfy the equation.

Identify the Graph of a Function

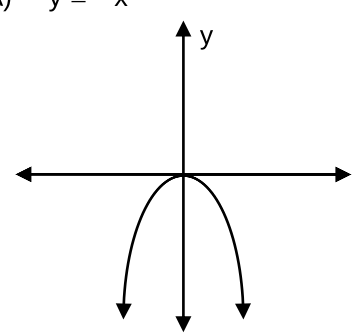
Not every collection of points in the x-y plane represents the graph of a function. Remember, for a function, each number *x* in the domain has exactly one image *y* in the range. This means that the graph of a function cannot contain two points with the same x-coordinate and different y-coordinates. The graph of a function must satisfy the **vertical-line test.**

Theorem: Vertical-Line Test

A set of points in the xy-plane is the graph of a function if and only if every vertical line intersects the graph in at most one point.

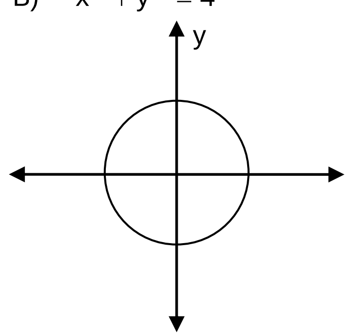
Example 1: Which of the graphs below are graphs of functions? Why or why not?

A) $y = -x^2$



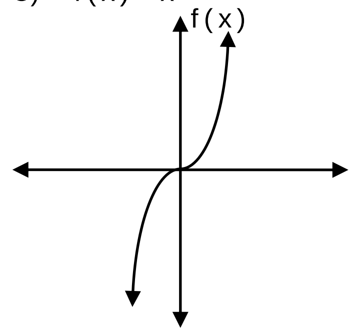
Function: Yes or No ?
Circle: Passes or Fails the Vertical-Line Test

B) $x^2 + y^2 = 4$



Function: Yes or No ?
Passes or Fails the Vertical-Line Test

C) $f(x) = x^3$



Function: Yes or No ?
Passes or Fails the Vertical-Line Test

Section 2.2 – The Graph of a Function (continued)

Obtain Information from or about the Graph of a Function

If (x, y) is a point on the graph of a function f, then y is the value of f at x; that is, $y = f(x)$. Also, if $y = f(x)$, then (x, y) is a point on the graph of f. For example, if $(-3,8)$ is on the graph of f, then $f(-3) = 8$, and if $f(6) = 9$, then the point $(6,9)$ is on the graph of $y = f(x)$. The next example illustrates how to obtain information about a function if its graph is given.

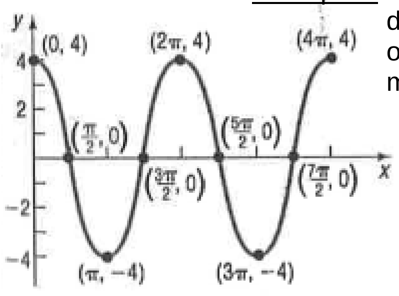


Figure 15

Example 2: Let f be the function whose graph is given in Figure 15. The graph of f may represent the distance y that the bob of a pendulum is from its at-rest position at time x. Negative values of y mean that the pendulum is to the left of the at-rest position, and positive values of y mean that the pendulum is to the right of the at-rest position.

A) What are $f(0), f\left(\frac{3}{2}\right)$, and $f(3\pi)$?

Since $(0,4)$ is on the graph of f, $f(0) = 4$. Similarly, when $x = \frac{3\pi}{2}$, $y = 0$, so $f\left(\frac{3}{2}\right) = 0$. When $x = 3\pi$, $y = -4$, so $f(3\pi) = -4$.

B) What is the domain of f ?

The points on the graph have x-coordinates between 0 and 4π, inclusive; and for each number x between 0 and 4π, there is a point $\left(x, f(x)\right)$ on the graph. The domain of f is $\{x | 0 \le x \le 4\pi\}$ or the interval $[0, 4\pi]$.

C) What is the range of f ?

The points on the graph all have y-coordinates between -4 and 4, inclusive; and for each such number y, there is at least one number x in the domain. The range of f is $\{y | -4 \le x \le 4\}$ or the interval $[-4, 4]$.

D) List the intercepts.

The x-intercepts are $\left(\frac{\pi}{2}, 0\right)$, $\left(\frac{3\pi}{2}, 0\right)$, $\left(\frac{5\pi}{2}, 0\right)$, and $\left(\frac{7\pi}{2}, 0\right)$.

The y-intercept is $(0,4)$.

E) How many times does the line $y = 2$ intersect the graph?

Draw the horizontal line $y = 2$ on the graph in Figure 15. The line intersects the graph four times.

F) For what values of x does $f(x) = -4$?

The only points on the graph with y-coordinates of -4 are $(\pi, -4)$ and $(3\pi, -4)$. So, $f(x) = -4$ when $x = \pi$ and $x = 3\pi$.

G) For what values of x is $f(x) > 0$?

Look at Figure 15 and determine the x-values from 0 to 4π for which the y-coordinate is positive.

$f(x) > 0$ for $0 \le x < \frac{\pi}{2}$, $\frac{3\pi}{2} < x < \frac{5\pi}{2}$, and $\frac{7\pi}{2} < x \le 4\pi$.

When the graph of a function is given, its domain may be viewed as the shadow created by the graph on the x-axis by vertical beams of light. Similarly, its range can be viewed as the shadow created by the graph on the y-axis by horizontal beams of light. Try this technique with the graph in Figure 15.

Section 2.2 – The Graph of a Function (continued)

You can also obtain information about the graph of a function by considering the function.

Example 3: Consider the function $f(x) = \dfrac{4}{x+3}$

A) Find the domain of f. The domain of f is $\{x | x \neq -3\}$.

B) Is the point $\left(1, \dfrac{4}{3}\right)$ on the graph of f ?

When $x = 1$, then

$$f(1) = \frac{4}{1+3} = \frac{4}{4} = 1$$

$\Rightarrow$ No, $(1, 1)$ is on the graph, not the point $\left(1, \dfrac{4}{3}\right)$.

C) If $x = -2$, what is $f(x)$? What point is on the graph of f ?

If $x = -2$, then $f(-2) = \dfrac{4}{-2+3}$

$$= \frac{4}{1} = 4$$

$\Rightarrow f(-2) = 4$, so $(-2, 4)$ is on the graph

D) If $f(x) = -2$, what is x ?
What point is on the graph of f?

$$\frac{4}{x+3} = f(x)$$

$$\frac{4}{x+3} = -2$$

$$4 = -2(x+3)$$

$$4 = -2x - 6$$

$$10 = -2x$$

$$x = -5$$

If $f(x) = -2$, then $x = -5$.
The point $(-5, -2)$ is on the graph of f.

E) What are the x-intercepts of the graph of f (if any)? What point(s) are on the graph of f?

The x-intercepts of the graph of f are the real solutions of the equation $f(x) = 0$ that are in the domain of f.

Set $f(x) = 0$.

$$\frac{4}{x+3} = 0$$

$4 = 0$ Multiply both sides by x+3

This is a false statement, so $f(x)$ has no x-intercept.

Summary

Graph of a Function The collection of points (x, y) that satisfies the equation $y = f(x)$.

Vertical Line Test A collection of points is the graph of a function if and only if every vertical line intersects the graph in at most one point.

All material has been taken from *Precalculus,* by M. Sullivan, 10th Edition