Section 2.3 – Properties of Functions – Day 1

To find the graph of a function $y = f(x)$, it is often helpful to know certain properties that the function has and the impact of these properties on the way the graph will look.

Determine Even and Odd Functions from a Graph

When applied to a function f, the words even and odd describe the symmetry that exists for the graph of the function.

A function f is even if and only if, whenever the point (x, y) is on the graph of f, the point $(-x, y)$ is also on the graph. Using function notation, we define an even function as follows:

A function f is **even** if, for every number x in its domain, the number $-x$ is also in the domain and $f(-x) = f(x)$.

A function f is odd if and only if, whenever the point (x, y) is on the graph of f, the point $(-x, -y)$ is also on the graph. Using function notation, we define an odd function as follows:

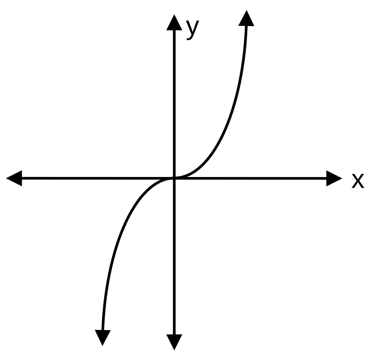
A function f is **odd** if, for every number x in its domain, the number $-x$ is also in the domain and $f(-x) = -f(x)$.

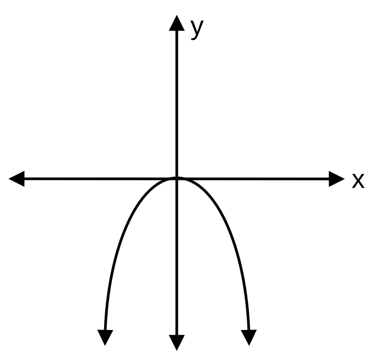
Go back and refresh your memory on the symmetry tests you learned in Section 1.2.

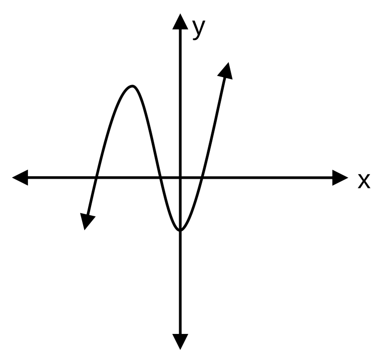
Theorem: A function is even if and only if its graph is symmetric with respect to the y-axis. A function is odd if and only if its graph is symmetric with respect to the origin.

You should be able to determine, both graphically and algebraically, whether functions are even, odd, or neither.

Example 1: Determine whether the graphed functions are even, odd, or neither.







Circle: Even, Odd, or Neither? If even or odd: sym wrt ____________

Even, Odd, or Neither? If even or odd: sym wrt ____________

Even, Odd, or Neither? If even or odd: sym wrt ________

Identify Even and Odd Functions from an Equation

Example 2: Determine algebraically whether the functions are even, odd, or neither. Then determine whether the graph is symmetric with respect to the y-axis, with respect to the origin, or neither.

To determine whether each function is even, odd, or neither, first replace x with $-x$.

A) $f(x) = x^3 + 2x^2$

$f(-x) = (-x)^3 + 2(-x)^2$

$= -x^3 + 2x^2$

$\Rightarrow f(-x) \neq f(x)$

$\Rightarrow$ Not an Even function

Now, $-f(x) = -x^3 - 2x^2$

$\Rightarrow f(-x) \neq -f(x)$

$\Rightarrow$ Not an Odd function

So,it is neither Even nor Odd

B) $g(x) = 2x^3$

$g(-x) = 2(-x)^3$

$= -2x^3$

So, $g(-x) \neq g(x)$

$\Rightarrow$ Not an Even function

Now, $-g(x) = -2x^3$

$\Rightarrow g(-x) = -g(x)$

$\Rightarrow$ Odd function, symmetric wrt the origin

C) $f(x) = 4x^2 + 2$

$f(-x) = 4(-x)^2 + 2$

$= 4x^2 + 2$

$\Rightarrow f(-x) = f(x)$

$\Rightarrow$ Even function,

symmetric wrt the y-axis

Section 2.3 – Properties of Functions – Day 1 (continued)

Use a Graph to Determine Where a Function is Increasing, Decreasing or Constant

Consider the graph given in Figure 18 below. If you look from left to right along the graph of the function, you will notice that parts of the graph are going up, parts are going down, and parts are horizontal. In such cases, the function is described as *increasing*, *decreasing*, or *constant*, respectively.

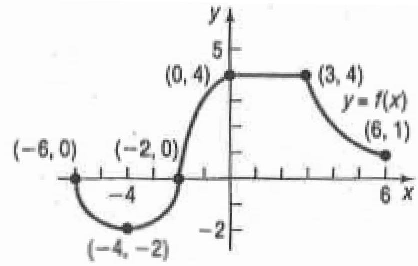


Figure 18

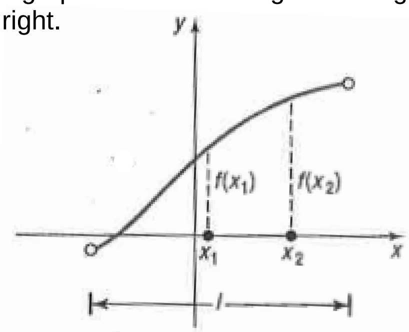
Describe the behavior of a graph in terms of its x-values. Use strict inequalities involving the independent variable x or use open intervals of x-coordinates when determining where a function is increasing, decreasing, or constant.

Example 3: Determine the values of x for which the function in Figure 18 above is increasing. Where is it decreasing, and where is it constant?

The function in Figure 18 above is increasing on the open interval $(-4,0)$, or for $-4 < x < 0$. The function is decreasing on the open intervals $(-6,-4)$ and $(3,6)$, or for $-6 < x < -4$ and $3 < x < 6$. The function is constant on the open interval $(0,3)$, or for $0 < x < 3$.

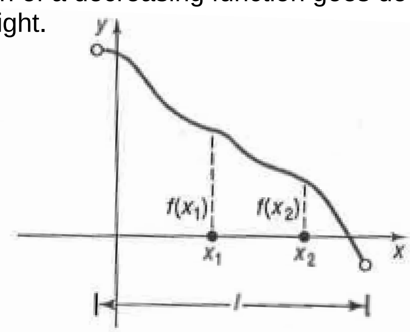
Definitions of Increasing and Decreasing Functions:

A function f is increasing on an open interval I, if for any choice of x_1 and x_2 in I, with $x_1 < x_2$, $f(x_1) < f(x_2)$. So, the graph of an increasing function goes up from left to right.



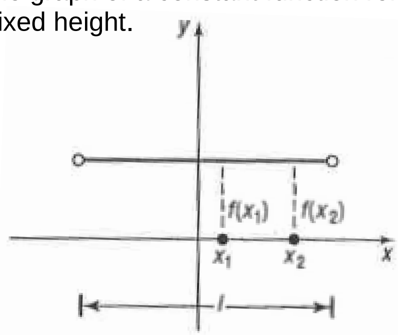
(a) For $x_1 < x_2$ in I, $f(x_1) < f(x_2)$; f is increasing on I.

A function f is decreasing on an open interval I, if for any choice of x_1 and x_2 in I, with $x_1 < x_2$, $f(x_1) > f(x_2)$. So, the graph of a decreasing function goes down from left to right.



(b) For $x_1 < x_2$ in I, $f(x_1) > f(x_2)$; f is decreasing on I.

A function f is constant on an open interval I, if for all choices of x in I, the values $f(x)$ are equal. So, the graph of a constant function remains at a fixed height.



(c) For all x in I, the values of f are equal; f is constant on I.

If a function is decreasing, then as the values of x get bigger, the values of the function get smaller.

If a function is increasing, then as the values of x get bigger, the values of the function also get bigger.

If a function is constant, then as the values of x get bigger, the values of the function remain unchanged.

Section 2.3 – Properties of Functions – Day 1 (continued)

Use a Graph to Locate Local Maxima and Local Minima

Suppose f is a function defined on an open interval I containing c. If the value of f at c is greater than or equal to the values of f on I, then f has a local maximum at c. See Figure 20(a) below.

If the value of f at c is less than or equal to the values of f on I, then f has a local minimum at c. See Figure 20(b) below.

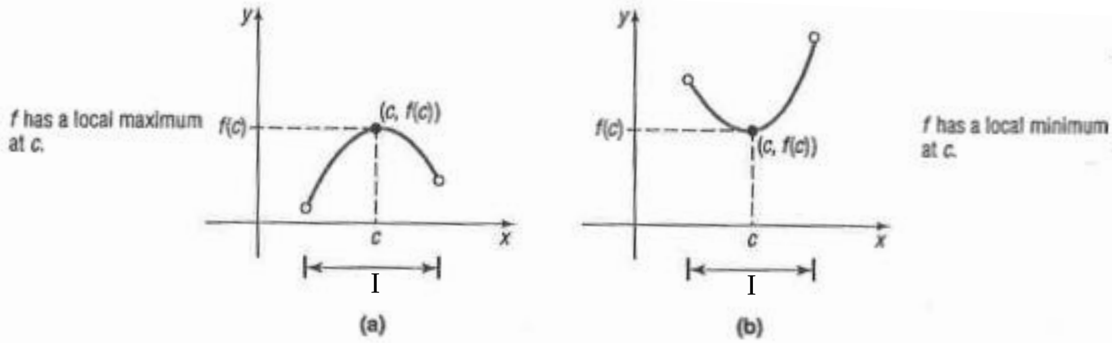


Figure 20 Local maximum and local minimum

Let f be a function defined on some open interval I.

A function f has a **local maximum** at *c* if there is an open interval I containing *c* so that, for all *x* in this open interval, you have $f(x) \le f(c)$. We call $f(c)$ a **local maximum value of f**. (Some texts use "relative" rather than "local.")

A function f has a **local minimum** at c if there is an open interval I containing *c* so that, for all *x* in this open interval, you have $f(x) \ge f(c)$. We call $f(c)$ a **local minimum value of f**.

If f has a local maximum at *c*, then the value of f at *c* is greater than or equal to the values of f near *c*. If f has a local minimum at *c*, then the value of f at *c* is less than or equal to the values of f near *c*. The word local is used to suggest that it is only near *c*, not necessarily over the entire domain, that the value $f(c)$ has these properties.

To locate the exact value at which a function f has a local maximum or a local minimum usually requires calculus. But, a graphing calculator may be used to approximate these values by using the calculator's MAXIMUM and MINIMUM features. (TI-83: 2nd Trace (Calc), maximum or minimum)

Example 4: Finding Local Maxima and Local Minima from the Graph of a Function and Determining Where the Function is Increasing, Decreasing, or Constant

Figure 21 shows the graph of a function f.

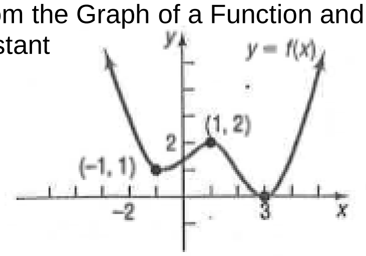


Figure 21

A) At what value(s) of x, if any, does f have a local maximum? List the local maximum values.

f has a local maximum at 1, since for all x close to 1, you have $f(x) \le f(1)$. The local maximum value is $f(1) = 2$.

B) At what value(s) of x, if any, does f have a local minimum? List the local minimum values.

f has local minima at -1 and at 3. The local minimum values are $f(-1) = 1$ and $f(3) = 0$.

C) Find the intervals on which f is increasing. Find the intervals on which f is decreasing.

The function in figure 21 is increasing for all values of x between -1 and 1 and for all values of x greater than 3. That is, the function is increasing on the intervals $(-1,1)$ and $(3,\infty)$, or for $-1 < x < 1$ and $x > 3$. The function is decreasing for all values of x less than -1 and for all values of x between 1 and 3. That is the function is decreasing on the intervals $(-\infty,-1)$ and $(1,3)$ or for $x < -1$ and $1 < x < 3$.

Section 2.3 – Properties of Functions – Day 1 (continued)

The y-value is the local maximum value or local minimum value, and it occurs at some x-value. For example, in Figure 21, you say f has a local maximum at 1 and the local maximum value is 2.

Use a Graph to Locate the Absolute Maximum and the Absolute Minimum

Look at the graph of the function f given in Figure 22.
The domain of f is the closed interval $[a,b]$. Also, the largest value of f is $f(u)$ and the smallest value of f is $f(v)$.
These are called, respectively, the *absolute maximum* and the *absolute minimum* of f on $[a,b]$.

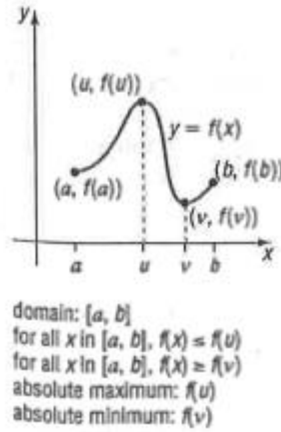


Figure 22

Definition:

Let f be a function defined on some open interval I. If there is a number *u* in I for which $f(x) \le f(u)$ for all x in I, then f has an **absolute maximum at u**, and the number $f(u)$ is the **absolute maximum of f on I**.

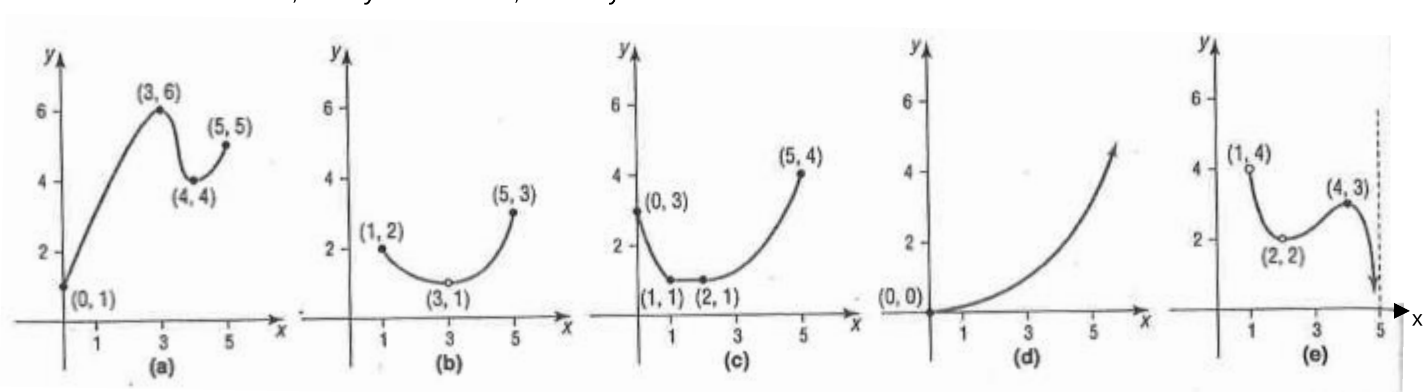
If there is a number v in I for which $f(x) \ge f(v)$ for all x in I, then f has an **absolute minimum at v**, and the number $f(v)$ is the **absolute minimum of f on I**.

The absolute maximum and absolute minimum of a function f are sometimes called the **extreme values** of f on I.

The absolute maximum or absolute minimum of a function may not exist.

Warning: A function may have an absolute maximum or an absolute minimum at an endpoint but not a local maximum or a local minimum. Why? Local maxima and local minima are found over some open interval I, and this interval cannot be created around an endpoint.

Example 5: For each graph of a function $y = f(x)$ in the figure below, find the absolute maximum and the absolute minimum, if they exist. Also, find any local maxima or local minima.



a) The function f whose graph is given in figure a) above has the closed interval $[0,5]$ as its domain. The largest value of f is $f(3) = 6$, the absolute maximum. The smallest value of f is $f(0) = 1$, the absolute minimum. The function has a local maximum of 6 at $x = 3$ and a local minimum of 4 at $x = 4$.

Section 2.3 – Properties of Functions – Day 1 (continued)

b) The function f whose graph is given in figure b) above has the domain $\{x \mid 1 \le x \le 5, x \ne 3\}$. Note that you exclude 3 from the domain because of the "hole" at $(3,1)$. The largest value of f on its domain is $f(5) = 3$, the absolute maximum. There is no absolute minimum. As you trace the graph, getting closer to the point $(3,1)$, there is no single smallest value. The function has no local maxima or minima.

c) The function f whose graph is given in figure c) above has the interval $[0,5]$ as its domain. The absolute maximum of f is $f(5) = 4$. The absolute minimum is 1. Notice that the absolute minimum 1 occurs at any number in the interval $[1,2]$. The function has a local minimum value of 1 at every x in the interval $[1,2]$, but it has no local maximum value.

d) The function f given in figure d) above has the interval $[0,\infty)$ as its domain. The function has no absolute maximum; the absolute minimum is $f(0) = 0$. The function has no local maximum or local minimum.

e) The function f given in figure d) above has the domain $\{x \mid 1 < x < 5, x \ne 2\}$. The function has no absolute maximum and no absolute minimum. The function has a local maximum value of 3 at $x = 4$, but no local minimum value.

In calculus, there is a theorem with conditions that guarantee a function will have an absolute maximum and an absolute minimum.

Theorem: Extreme Value Theorem

If f is a continuous function whose domain is a closed interval $[a,b]$, then f has an absolute maximum and an absolute minimum.

Since a precise definition of continuous requires calculus, we will assume that a continuous function is one whose graph has no holes or gaps and can be traced without lifting your pencil from the paper.

The absolute maximum (minimum) can be found by selecting the largest (smallest) value of f from the following list:

1. The values of f at any local maxima or local minima of f in $[a,b]$.
2. The value of f at each endpoint of $[a,b]$ – that is, $f(a)$ and $f(b)$.

For example, the graph of the function f in figure a) above is continuous on the closed interval $[0,5]$. The Extreme Value Theorem guarantees that f has extreme values on $[0,5]$. To find them, list

1. The value of f at the local extrema: $f(3) = 6$ and $f(4) = 4$.
2. The value of f at the endpoints: $f(0) = 1$ and $f(5) = 5$.

The largest of these, 6, is the absolute maximum; the smallest of these, 1, is the absolute minimum.

Section 2.3 – Properties of Functions – Day 1 (continued)

Use a Graphing Utility to Approximate Local Maxima and Local Minima and to Determine Where a Function is Increasing or Decreasing

Example 6: A) Use a graphing utility to graph the function $f(x) = 6x^3 - 12x + 5$ for $-2 < x < 2$. Approximate where f has a local maximum and where f has a local minimum.

B) Determine where f is increasing and where it is decreasing.

Solution:

A) Graphing utilities have a feature that finds the maximum or minimum point of a graph within a given interval. Graph the function f for $-2 < x < 2$. The MAXIMUM and MINIMUM commands require you to first determine the open interval I. The graphing utility will then approximate the maximum or minimum value in the interval. Using MAXIMUM, you will find the local maximum value is 11.53 and that it occurs at $x = -0.82$, rounded to two decimal places. See Figure 24(a) below.

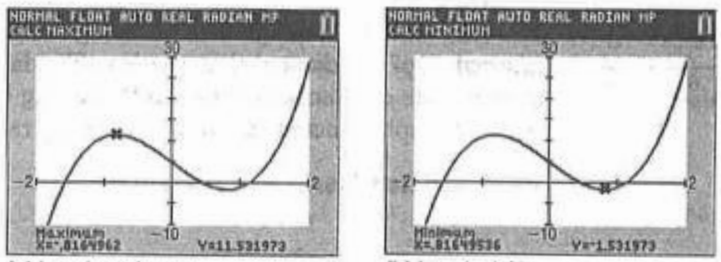


Figure 24 (a) Local maximum (b) Local minimum

Using MINIMUM, you will find that the local minimum value is -1.53 and it occurs at $x = 0.82$, rounded to two decimal places. See Figure 24(b) above.

B) Looking at Figures 24 (a) and (b), you can see that the graph of f is increasing from $x = -2$ to $x = -0.82$ and from $x = 0.82$ to $x = 2$, so f is increasing on the intervals $(-2, -0.82)$ and $(0.82, 2)$, or for $-2 < x < -0.82$ and $0.82 < x < 2$.

The graph is decreasing from $x = -0.82$ to $x = 0.82$, so f is decreasing on the interval $(-0.82, 0.82)$, or for $-0.82 < x < 0.82$.