Section 2.3 – Properties of Functions – Day 2

Find the Average Rate of Change of a Function

In Section 1.3, we said that the slope of a line can be interpreted as the average rate of change. To find the average rate of change of a function between any two points on its graph, calculate the slope of the line containing the two points.

If a and b, $a \neq b$, are in the domain of a function $y = f(x)$, the **average rate of change of f** from a to b is defined as

$$\text{Average Rate of Change} = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}, \quad a \neq b.$$

The symbol Δ is the Greek capital letter delta and it is read as "change in." The symbol Δy in the equation above is the "change in y," and Δx is the "change in x." The average rate of change of f is the change in y divided by the change in x.

Example 7: Find the Average Rate of Change

Find the average rate of change of $f(x) = 3x^2$.

a) From 1 to 3 b) From 1 to 5 c) From 1 to 7

a) The average rate of change of $f(x) = 3x^2$ from 1 to 3 is

$$\frac{\Delta y}{\Delta x} = \frac{f(3) - f(1)}{3 - 1} = \frac{3(3)^2 - 3(1)^2}{3 - 1} = \frac{3(9) - 3(1)}{2} = \frac{27 - 3}{2} = \frac{24}{2} = 12$$

b) The average rate of change of $f(x) = 3x^2$ from 1 to 5 is

$$\frac{\Delta y}{\Delta x} = \frac{f(5) - f(1)}{5 - 1} = \frac{3(5)^2 - 3(1)^2}{5 - 1} = \frac{3(25) - 3(1)}{4} = \frac{75 - 3}{4} = \frac{72}{4} = 18$$

c) The average rate of change of $f(x) = 3x^2$ from 1 to 7 is

$$\frac{\Delta y}{\Delta x} = \frac{f(7) - f(1)}{7 - 1} = \frac{3(7)^2 - 3(1)^2}{7 - 1} = \frac{3(49) - 3(1)}{6} = \frac{147 - 3}{6} = \frac{144}{6} = 24$$

Figure 25 is a graph of $f(x) = 3x^2$.

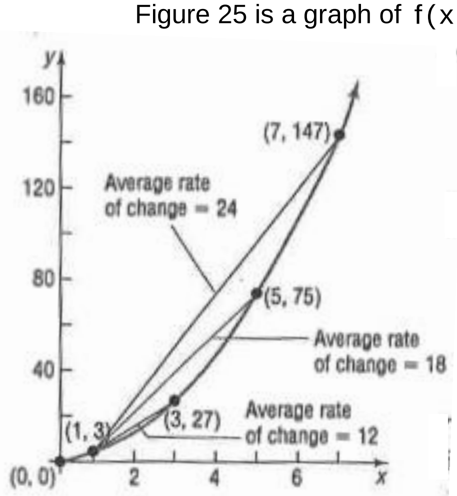
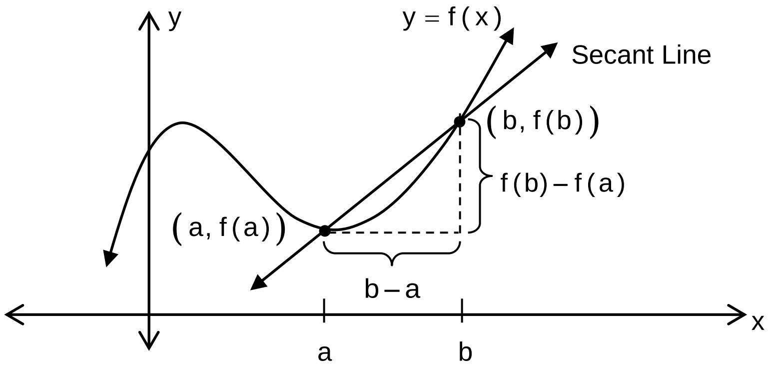


Figure 25 $f(x) = 3x^2$

You can see that the function f is increasing for $x > 0$. The fact that the average rate of change is positive for any $x_1, x_2, x_1 \neq x_2$, in the interval $(1, 7)$ indicates that the graph is increasing on $1 < x < 7$. Further, the average rate of change is consistently getting larger for $1 < x < 7$, which indicates that the graph is increasing at an increasing rate.

Section 2.3 – Properties of Functions – Day 2 (continued)

The average rate of change of a function has an important geometric interpretation. Look at the graph of $y = f(x)$ in the figure below. Two points are labeled on the graph: $(a, f(a))$ and $(b, f(b))$. The line containing these two points is called the secant line; its slope is $m_{sec} = \dfrac{f(b) - f(a)}{b - a}$.



Theorem: Slope of a Secant Line

> The average rate of change of a function from *a* to *b* equals the slope of the secant line containing the two points $(a, f(a))$ and $(b, f(b))$ on its graph.

Example 8: Finding the Equation of a Secant Line

Suppose that $g(x) = 3x^2 - 2x + 3$.

a) Find the average rate of change of g from -2 to 1.

$$\text{The average rate of change } = \frac{g(1) - g(-2)}{1 - (-2)}$$
$$= \frac{(3(1)^2 - 2(1) + 3) - (3(-2)^2 - 2(-2) + 3)}{1 - (-2)}$$
$$= \frac{(3(1) - 2 + 3) - (3(4) + 4 + 3)}{1 + 2}$$
$$= \frac{(3 + 1) - (12 + 7)}{3}$$
$$= \frac{4 - 19}{3}$$
$$= \frac{-15}{3}$$
$$= -5$$

b) Find an equation of the secant line containing the points $(-2, g(-2))$ and $(1, g(1))$.

From a) $(-2, g(-2)) = (-2, 19)$ and $(1, g(1)) = (1, 4)$

$y - y_1 = m_{sec}(x - x_1)$

$y - 19 = -5(x - (-2))$

$y - 19 = -5(x + 2)$

$y - 19 = -5x - 10$

$y = -5x + 9$

c) Using a calculator, draw the graph of g and the secant line obtained in part b) on the same graph.

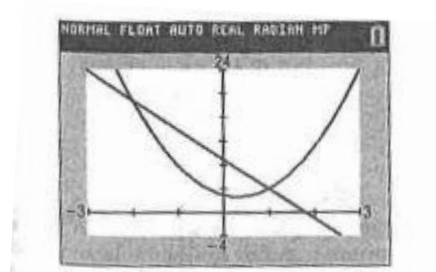


Figure 27 Graph of g and the secant line

Section 2.3 – Properties of Functions – Day 2 (continued)

Example 9: The height of a marker thrown from a hot-air balloon is given by the function $h(t) = -16t^2 - 30t + 200$, where h is in feet. Find the average rate of change of the height of the marker between 2 and t seconds.

$$\begin{aligned}
\frac{\Delta h}{\Delta t} &= \frac{h(t) - h(2)}{t-2}, \text{ where } t \neq 2. \\
&= \frac{(-16t^2 - 30t + 200) - (-16(2)^2 - 30(2) + 200)}{t-2} \\
&= \frac{(-16t^2 - 30t + 200) - (-64 - 60 + 200)}{t-2} \\
&= \frac{(-16t^2 - 30t + 200) - 76}{t-2} \\
&= \frac{-16t^2 - 30t + 124}{t-2} \\
&= \frac{(-16t - 62)\cancel{(t-2)}}{\cancel{t-2}} \\
&= -16t - 62
\end{aligned}$$

So, the average rate of change of the height of the marker between 2 and t seconds is $\frac{h(t) - h(2)}{t-2} = -16t - 62$.