Section 3.1 – Properties of Linear Functions and Linear Models

Graph Linear Functions

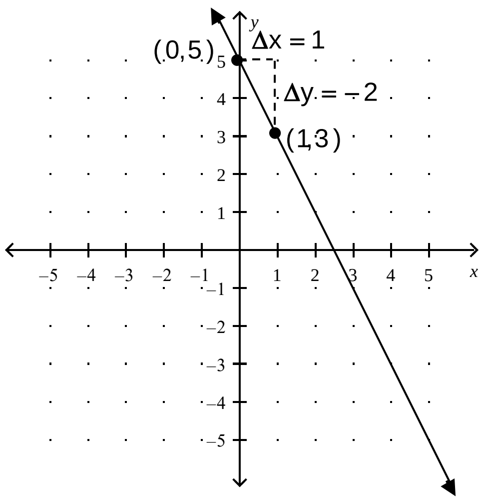
In Section 1.3 we discussed lines. In particular, for non-vertical lines we developed the slope-intercept form of the equation of a line $y = mx + b$. When the slope-intercept form of a line is written using function notation, the result is a *linear function*.

A **linear function** is a function of the form $f(x) = mx + b$.

The graph of a linear function is a line with slope *m* and y-intercept *b*. Its domain is the set of all real numbers.

Functions that are not linear are said to be **nonlinear**.

Example 1: Graph the linear function $f(x) = -2x + 5$. What are the domain and the range of f ?



This is a linear function with slope $m = -2$ and y-intercept $b = 5$.

To graph this function, plot the point $(0,5)$, the y-intercept, and use the slope to find an additional point by moving right 1 unit and down 2 units.

The domain and the range of f are the set of all real numbers.

Use Average Rate of Change to Identify Linear Functions

Table 1:

x	$f(x) = -2x + 5$	Average Rate of Change $\frac{\Delta y}{\Delta x}$
-2	9	
		$\frac{7-9}{-1-(-2)} = \frac{-2}{1} = -2$
-1	7	
		$\frac{5-7}{0-(-1)} = \frac{-2}{1} = -2$
0	5	
		$\frac{3-5}{1-0} = \frac{-2}{1} = -2$
1	3	

Look at Table 1, which shows certain values of the independent variable *x* and corresponding values of the dependent variable *y* for the function $f(x) = -2x + 5$. Notice that as the value of the independent variable, *x*, increases by 1, the value of the dependent variable *y* decreases by 2. That is, the average rate of change of *y* with respect to *x* is a constant, -2.

It is not a coincidence that the average rate of change of the linear function $f(x) = -2x + 5$ is the slope of the linear function. That is, $\frac{\Delta y}{\Delta x} = m = -2$. The following theorem states this fact.

Theorem: Average Rate of Change

Linear functions have a constant average rate of change. That is, the average rate of change of a linear function $f(x) = mx + b$ is $\frac{\Delta y}{\Delta x} = m$.

Section 3.1 – Properties of Linear Functions and Linear Models (continued)

Proof of the Average Rate of Change Theorem:

The average rate of change of $f(x)=mx+b$ from x_1 to $x_2, x_1 \neq x_2$, is

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{f(x_2)-f(x_1)}{x_2-x_1}\\ &= \frac{(mx_2+b)-(mx_1+b)}{x_2-x_1}\\ &= \frac{mx_2-mx_1}{x_2-x_1}\\ &= \frac{m(x_2-x_1)}{x_2-x_1}\\ &= m\end{aligned}$$

As it turns out, only linear functions have a constant average rate of change. Because of this, the average rate of change can be used to determine whether a function is linear. This is especially useful if the function is defined by a data set.

Example 2: Using the Average Rate of Change to Identify Linear Functions

a) A strain of *E. coli* known as Beu 397-recA441 is placed into a Petri dish at 30° Celsius and allowed to grow. The data shown in Table 2 below are collected. The population is measured in grams and the time in hours. Plot the ordered pairs (x,y) in the Cartesian plane, and use the average rate of change to determine whether the function is linear.

b) The data in Table 3 represent the maximum number of heartbeats that a healthy individual of different ages should have during a 15-second interval of time while exercising. Plot the ordered pairs (x,y) in the Cartesian plane, and use the average rate of change to determine whether the function in linear.

Table 2

Time (hours), x	Population (grams), y	(x, y)
0	0.09	(0, 0.09)
1	0.12	(1, 0.12)
2	0.16	(2, 0.16)
3	0.22	(3, 0.22)
4	0.29	(4, 0.29)
5	0.39	(5, 0.39)

Table 3

Age, x	Maximum Number of Heartbeats, y	(x, y)
20	50	(20, 50)
30	47.5	(30, 47.5)
40	45	(40, 45)
50	42.5	(50, 42.5)
60	40	(60, 40)
70	37.5	(70, 37.5)

Source: American Heart Association

Solution: Compute the average rate of change of each function. If the average rate of change is constant, the function is linear. If the average rate of change is not constant, the function is not linear.

a) Figure 2 shows the points listed in Table 2 plotted in the Cartesian plane. Note that it is impossible to draw a straight line that contains all the points. Table 4 displays the average rate of change of the population.

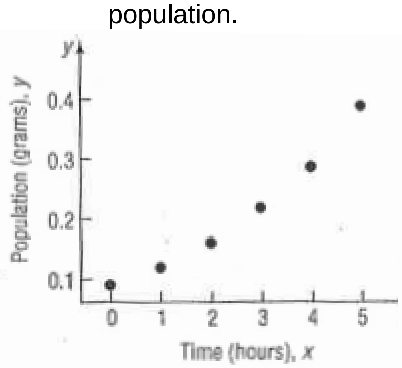


Figure 2

Table 4

Time (hours), x	Population (grams), y	Average Rate of Change $= \frac{\Delta y}{\Delta x}$
0	0.09	
		$\frac{0.12-0.09}{1-0} = 0.03$
1	0.12	
		0.04
2	0.16	
		0.06
3	0.22	
		0.07
4	0.29	
		0.10
5	0.39	

Section 3.1 – Properties of Linear Functions and Linear Models (continued)

Because the average rate of change is not constant, the function is not linear. In fact, because the average rate of change is increasing as the value of the independent variable increases, the function is increasing at an increasing rate. So not only is the population increasing over time, but it is also growing more rapidly as time passes.

b) Figure 3 shows the points listed in Table 3 plotted in the Cartesian plane. Note that the data in Figure 3 lie on a straight line. Table 5 displays the average rate of change of the maximum number of heartbeats. The average rate of change of the heartbeat data is constant, -0.25 beat per year, so the function is linear.

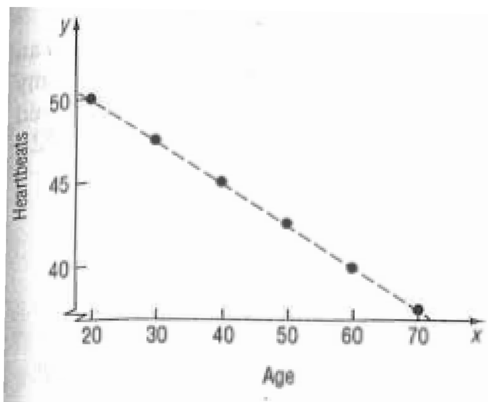


Figure 3

Table 5

Age, x	Maximum Number of Heartbeats, y	Average Rate of Change $= \frac{\Delta y}{\Delta x}$
20	50	
		$\frac{47.5 - 50}{30 - 20} = -0.25$
30	47.5	
		-0.25
40	45	
		-0.25
50	42.5	
		-0.25
60	40	
		-0.25
70	37.5	

Determining Whether a Linear Function is Increasing, Decreasing, or Constant

Example 3: Determine whether the following linear functions are increasing, decreasing, or constant.

a) $f(x) = 7x - 3$ b) $g(x) = -3x + 7$ c) $s(t) = \frac{-4}{5}t - 5$ d) $h(z) = 9$

a) For the linear function $f(x) = 7x - 3$, the slope is 7, which is positive. The function f is increasing on the interval $(-\infty, \infty)$.

b) For the linear function $g(x) = -3x + 7$, the slope is -3, which is negative. The function g is decreasing on the interval $(-\infty, \infty)$.

c) For the linear function $s(t) = \frac{-4}{5}t - 5$ the slope is $\frac{-4}{5}$, which is negative. The function s is decreasing on the interval $(-\infty, \infty)$.

d) The linear function h can be written as $h(z) = 0z + 9$. Because the slope is 0, the function h is constant on the interval $(-\infty, \infty)$.

Build Linear Models from Verbal Descriptions

When the average rate of change of a function is constant, a linear function can model the relation between the two variables. For example, if a recycling company pays $0.52 per pound for aluminum cans, then the relation between the price *p* paid and the pounds recycled *x* can be modeled as the linear function $p(x) = 0.52x$, with slope $m = \frac{0.52 \text{ dollar}}{\text{pound}}$.

Section 3.1 – Properties of Linear Functions and Linear Models (continued)

Modeling with a Linear Function

If the average rate of change of a function is a constant *m*, a linear function f can be used to model the relation between the two variables as follows: $f(x) = mx + b$, where *b* is the value of f at 0; that is, $b = f(0)$.

Example 4: Straight-line Depreciation

Book value is the value of an asset that a company uses to create its balance sheet. Some companies depreciate assets using straight-line depreciation so that the value of the asset declines by a fixed amount each year. The amount of the decline depends on the useful life that the company assigns to the asset. Suppose a company just purchased a fleet of new cars for its sales force at a cost of $31,500 per car. The company chooses to depreciate each vehicle using the straight-line method over 7 years. This means that each car will depreciate by $\frac{\$31,500}{7} = \$4,500$ per year.

a) Write a linear function that expresses the book value V of each car as a function of its age, x, in years.
b) Graph the linear function.
c) What is the book value of each car after 3 years?
d) Interpret the slope.
e) When will the book value of each car be $9,000? (Hint: Solve the equation $V(x) = 9000$.)

Solution: a) If you let $V(x)$ represent the value of each car after x years, then $V(0)$ represents the original value of each car, so $V(0) = \$31,500$. The y-intercept of the linear function is $31,500. Because each car depreciates by $4,500 per year, the slope of the linear function is $-4,500$. The linear function that represents the book value V of each car after x years is $V(x) = -4,500x + 31,500$.

Figure 4 shows the graph of V.

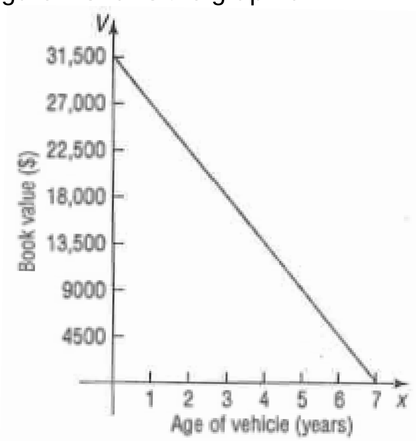


Figure 4 $V(x) = -4500x + 31,500$

The book value of each car after 3 years is

$$\begin{aligned} V(3) &= -4,500(3) + 31,500 \\ &= -13,500 + 31,500 \\ &= \$18,500 \end{aligned}$$

Since the slope of $V(x) = -4,500x + 31,500$ is $-4,500$, the average rate of change of the book value is $-4,500$ / year. So for each additional year that passes, the book value of the car decreases by $4,500.

To find when the book value will be $9,000, solve the equation

$$V(x) = 9000$$
$$-4,500x + 31,500 = 9000$$
$$-4,500x = -22,500$$
$$x = \frac{-22,500}{-4,500}$$
$$x = 5$$

The car will have a book value of $9,000 when it is 5 years old.

Section 3.1 – Properties of Linear Functions and Linear Models (continued)

Example 5: Supply and Demand

The quantity supplied of a good is the amount of a product that a company is willing to make available for sale at a given price. The quantity demanded of a good is the amount of a product that consumers are willing to purchase at a given price. Suppose that the quantity supplied, S, and the quantity demanded, D, of cellular telephones each month are given by the following functions:

$S(p) = 60p - 900$

$D(p) = -15p + 2850$, where p is the price (in dollars) of the telephone.

a) The **equilibrium price** of a product is defined as the price at which quantity supplied equals quantity demanded. That is, the equilibrium price is the price at which $S(p) = D(p)$. Find the equilibrium price of cellular telephones. What is the **equilibrium quantity**, the amount demanded (or supplied) at the equilibrium price?

b) Determine the prices for which quantity supplied is greater than quantity demanded. That is, solve the inequality $S(p) > D(p)$.

c) Graph $S = S(p)$ and $D = D(p)$, and label the **equilibrium point**, the point of intersection of S and D.

Solution: a) To find the equilibrium price, solve the equation $S(p) = D(p)$.

$$S(p) = D(p)$$
$$60p - 900 = -15p + 2850$$
$$60p = -15p + 3750$$
$$75p = 3750$$
$$p = 50$$

The equilibrium price is \$50 per cellular phone. To find the equilibrium quantity, evaluate either $S(p)$ or $D(p)$ at $p = 50$.

$$S(50) = 60(50) - 900$$
$$S(50) = 3000 - 900$$
$$S(50) = 2100$$

The equilibrium quantity is 2100 cellular phones. At a price of \$50 per phone, the company will produce and sell 2100 phones each month and have no shortages or excess inventory.

b) The inequality $S(p) > D(p)$ is

$$60p - 900 > -15p + 2850$$
$$60p > -15p + 3750$$
$$75p > 3750$$
$$p > 50$$

If the company charges more than \$50 per phone, quantity supplied will exceed quantity demanded. In this case, the company will have excess phones in inventory.

c) Figure 5 shows the graphs of $S = S(p)$ and $D = D(p)$ with the equilibrium point labeled.

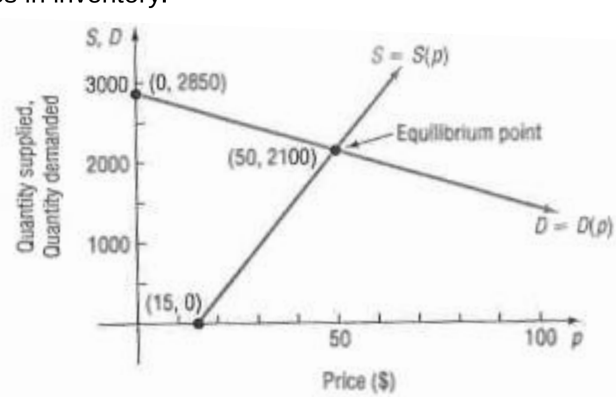


Figure 5 Supply and demand functions

All material has been taken from *Precalculus,* by M. Sullivan, 10th Edition