Section 3.3 – Quadratic Functions and Their Properties – Day 1

A quadratic function is a function of the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$. The domain of a quadratic function consists of all real numbers. The graph of a quadratic function is called a parabola.

Many applications require a knowledge of quadratic functions. For example, suppose that Texas Instruments collects the data shown in Table 7, which relate the number of calculators sold to the price *p* (in dollars) per calculator. Since the price of a product determines the quantity that will be purchased, we treat price as the independent variable. The relationship between the number *x* of calculators sold and the price *p* per calculator is given by the linear equation $x = 21{,}000 - 150p$.

Table 7

Price *p* per Calculator (in dollars)	Number of Calculators, *x*
60	12,000
65	11,250
70	10,500
75	9,750
80	9,000
85	8,250
90	7,500

Then the revenue R derived from selling *x* calculators at the price *p* per calculator is equal to the unit selling price *p* of the calculator times the number *x* of units actually sold. That is, $R = xp$

$$R(p) = (21{,}000 - 150p)p$$
$$= -150p^2 + 21{,}000p$$

So the revenue R is a quadratic function of the price *p*. Figure 12 illustrates the graph of this revenue function, whose domain is $0 \le p \le 140$, since both *x* and *p* must be nonnegative.

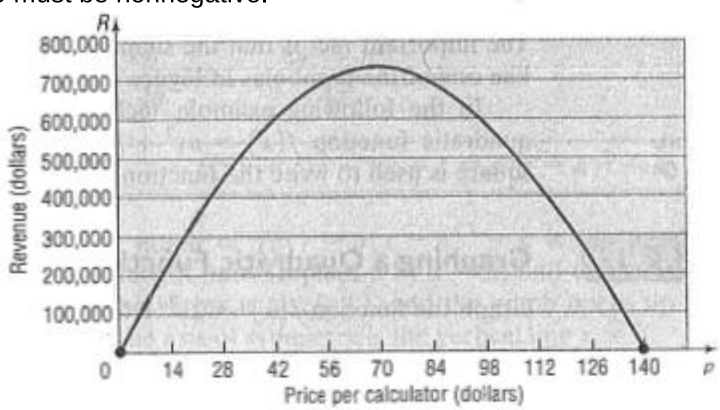


Figure 12
$R(p) = -150p^2 + 21{,}000p$

A second situation in which a quadratic function appears involves the motion of a projectile. Based on Newton's Second Law of Motion (force equals mass times acceleration, $F = ma$), it can be shown that, ignoring air resistance, the path of a projectile propelled upward at an inclination to the horizontal is the graph of a quadratic function. Figure 13 shows the path of a cannonball propelled upward.

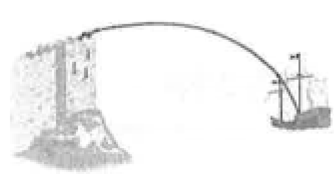


Figure 13
Path of a cannonball

Section 3.3 – Quadratic Functions and Their Properties – Day 1 (continued)

Graph a Quadratic Function Using Transformations

You know how to graph the square function $f(x)=x^2$. Figure 14 shows the graph of three functions of the form $f(x)=ax^2$, $a>0$, for $a=1$, $a=\frac{1}{2}$, and $a=3$. Note that the larger the value of *a*, the "narrower" the graph is, and the smaller the value of *a*, the "wider" the graph is.

Figure 15 shows the graphs of $f(x)=ax^2$ for $a<0$. Notice that these graphs are reflections about the x-axis of the graphs in Figure 14. Based on the results of these two figures, general conclusions can be drawn about the graph of $f(x)=ax^2$. First, as $|a|$ increases, the graph is stretched vertically (becomes "taller"), and as $|a|$ gets closer to zero, the graph is compressed vertically (becomes "shorter"). Second, if *a* is positive, the graph opens "up," and if *a* is negative, the graph opens "down."

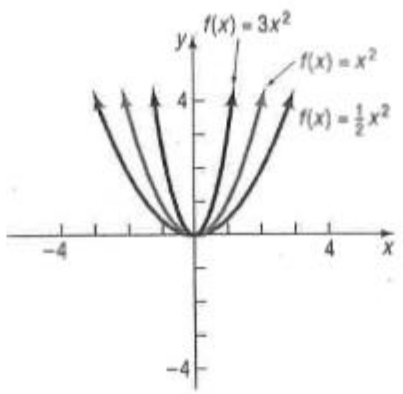


Figure 14

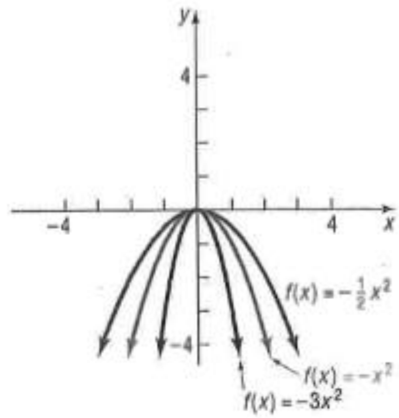


Figure 15

The graphs in Figures 14 and 15 are typical of the graphs of all quadratic functions, which are called **parabolas**. Refer to Figure 16, where two parabolas are pictured. The one on the left opens up and has a lowest point; the one on the right opens down and has a highest point. The lowest or highest point of a parabola is called the **vertex**. The vertical line passing through the vertex in each parabola in the Figure 16 is called the **axis of symmetry (AOS)** of the parabola. Because the parabola is symmetric about its axis, the axis of symmetry of a parabola can be used to find additional points on the parabola.

The parabolas shown in Figure 16 are the graphs of a quadratic function $f(x)=ax^2+bx+c, a\neq 0$. Notice that the coordinate axes are not included in the figure. Depending on the values of a, b, and c, the axes could be placed anywhere. The important fact is that the shape of the graph of a quadratic function will look like one of the parabolas in Figure 16.

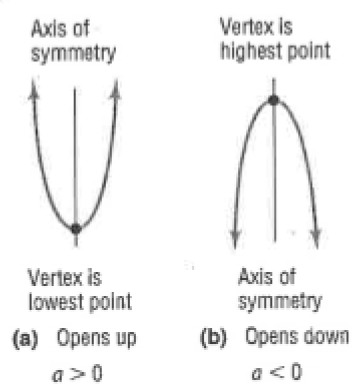


Figure 16
Graphs of a quadratic function, $f(x)=ax^2+bx+c,\ a\neq 0$

Section 3.3 – Quadratic Functions and Their Properties – Day 1 (continued)

Remember that you can use the method of completing the square to write the function $f(x) = ax^2 + bx + c$ in the form $f(x) = a(x-h)^2 + k$.

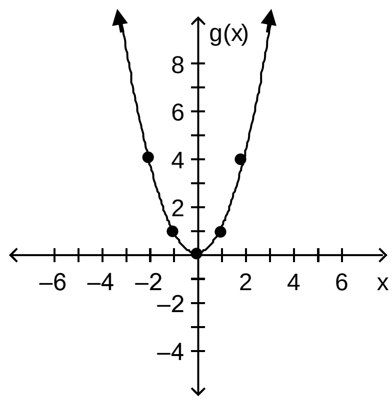
Example 1: Graph the quadratic function $f(x) = 2x^2 - 12x + 13$ using transformations.

$$\begin{aligned} f(x) &= 2x^2 - 12x + 13 \\ &= 2(x^2 - 6x) + 13 \\ &= 2(x^2 - 6x + 9) + 13 - 18 \quad \text{Complete the Square} \\ &= 2(x-3)^2 - 5 \end{aligned}$$

Transformations: **Label** the points graphed

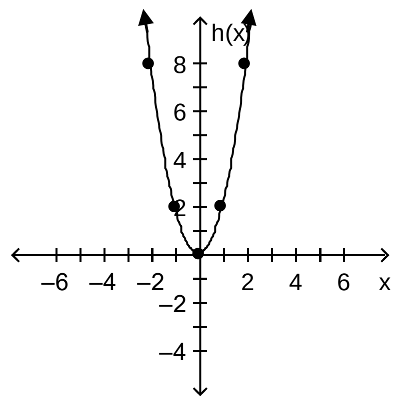
1) $g(x) = x^2$

Library/Basic function



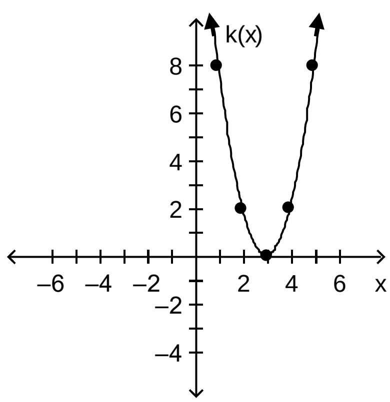
2) $h(x) = 2g(x)$
$= 2x^2$

Vertical stretch by a factor of 2



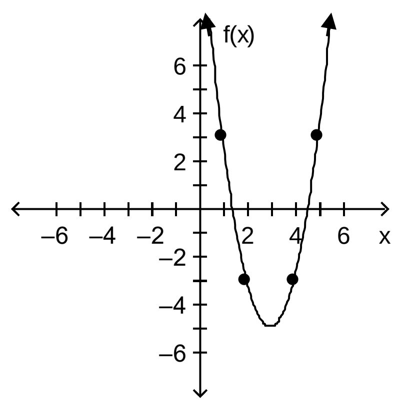
3) $k(x) = h(x-3)$
$= 2(x-3)^2$

Shift right 3 units



4) $f(x) = k(x) - 5$
$= 2(x-3)^2 - 5$

Shift down 5 units



Section 3.3 – Quadratic Functions and Their Properties – Day 1 (continued)

The method used in Example 1 can be used to graph any quadratic function $f(x)=ax^2+bx+c, a\neq 0,$ as follows:

$$f(x)=ax^2+bx+c$$

$$=a\left(x^2+\frac{b}{a}x\right)+c \qquad \text{Factor out } a \text{ from } ax^2+bx$$

$$=a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}\right)+c-a\left(\frac{b^2}{4a^2}\right) \qquad \text{Complete the square by adding } \frac{b^2}{4a^2} \text{ to the x terms}$$

$$=a\left(x+\frac{b}{2a}\right)^2+c-\frac{b^2}{4a} \qquad \text{Factor, simplify}$$

$$=a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a} \qquad \text{Common denominator}$$

These results lead to the following conclusion:

If $h=\frac{-b}{2a}$ and $k=\frac{4ac-b^2}{4a}$, then $f(x)=ax^2+bx+c$

$$=a(x-h)^2+k$$

The graph of $f(x)=a(x-h)^2+k$ is the parabola $g(x)=ax^2$ shifted horizontally *h* units (replace x by $x-h$) and vertically *k* units (add k). As a result, the vertex is at (h,k), and the graph opens up if $a>0$ and down if $a<0$. The axis of symmetry is the vertical line $x=h$.

So, in Example 1, we had $f(x)=2(x-3)^2-5$.

$$f(x)=a(x-h)^2+k$$

Because $a=2$, the graph opens upward. Also, because $h=3$ and $k=-5$, its vertex is at $(h,k)=(3,-5)$. Looking at Example 1 again, you can see that the shifts and final vertex agree with these statements.

All material has been taken from *Precalculus,* by M. Sullivan, 10th Edition