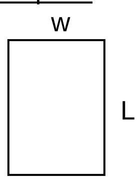
Section 3.4 – Build Quadratic Models from Verbal Descriptions

Mathematical models that involve functions are often used to solve real-life problems. The functions must be built based on the information given in the problem. You must be able to translate the verbal description into a mathematical model. Assigning symbols to represent the independent and dependent variables and then writing the function that relates these variables performs this translation.

When a mathematical model is in the form of a quadratic function, the properties of the graph of the function can provide important information about the model. In particular, you can use the quadratic function to determine the maximum or minimum value of the function. The fact that the graph of a quadratic function has a maximum or minimum value enables you to answer questions involving **optimization** – that is, finding the maximum or minimum values in models.

Build Quadratic Models from Verbal Descriptions

Example 1: The perimeter of a rectangle is 80 feet. Express its area A as a function of the side length L .



$$P = 2w + 2L \qquad \text{Area} = Lw$$
$$80 = 2w + 2L \qquad = L(40 - L)$$
$$40 = w + L \qquad = -L^2 + 40L$$
$$w = 40 - L$$

So, the area A as a function of side length L is $A(L) = -L^2 + 40L$

dependent variable (A) independent variable (L)

In economics, revenue R, in dollars, is defined as the amount of money received from the sale of an item and is equal to the unit selling price *p*, in dollars, of the item times the number *x* of units actually sold. That is, $R = xp$.

The Law of Demand states that *p* and *x* are related: As one increases, the other decreases. The equation that relates *p* and *x* is called the **demand equation**. When the demand equation is linear, the revenue model is a quadratic function.

Example 2: Suppose unit selling price p and the number x of units sold are related by the demand equation

$$p(x) = \frac{-1}{5}x + 40, \quad 0 \le x \le 300.$$

a) Express the revenue R as a function of the number x of units sold.

Since $R = xp$ and $p(x) = \frac{-1}{5}x + 40$, it follows that

$$R(x) = xp(x)$$
$$= x\left(\frac{-1}{5}x + 40\right)$$
$$= \frac{-1}{5}x^2 + 40x$$

b) Graph the revenue function $R(x)$. What quantity x maximizes the revenue? What is the maximum revenue?

It takes three points to graph a parabola. Here, use the maximum point and the x-intercepts.

How do you calculate the maximum point?

$$x_v = \frac{-b}{2a} = \frac{-40}{2\left(\frac{-1}{5}\right)} = \frac{-40}{-2/5} = \frac{-200}{-2} = 100$$

$$R(x_v) = R(100) = \frac{-1}{5}(100)^2 + 40(100) = \frac{-1}{5}(10000) + 4000 = -2000 + 4000 = 2000$$

So, the maximum is at $(100, 2000)$.

Find the x-intercepts:

Set $R(x) = 0$

$$\frac{-1}{5}x^2 + 40x = 0$$
$$x\left(\frac{-1}{5}x + 40\right) = 0$$
$$\Rightarrow x = 0 \quad \text{or} \quad \frac{-1}{5}x + 40 = 0$$
$$x - 200 = 0$$
$$x = 200$$

So, the x-intercepts are at $(0, 0)$ and $(200, 0)$.

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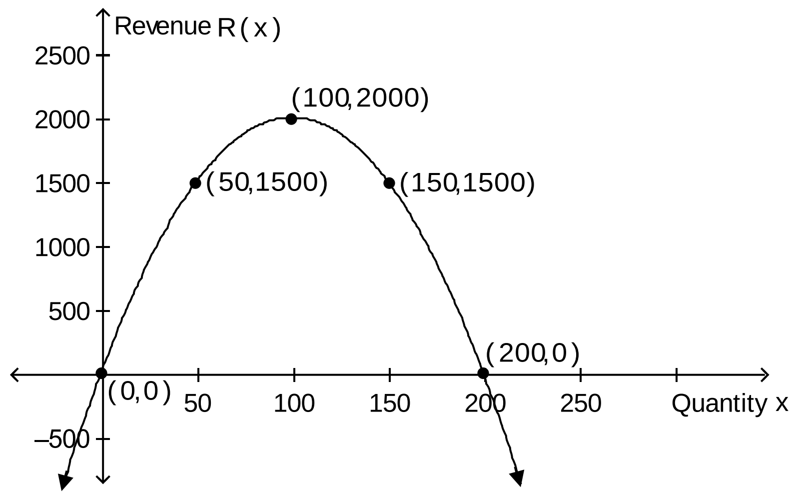

The maximum is at $(100, 2000)$. $\Rightarrow$ Selling 100 units will give you a maximum revenue of \$2000.

Example 3: The marketing department at Texas Instruments has found that when certain calculators are sold at a price of *p* dollars per unit, the number *x* of calculators sold is given by the demand equation $x(p) = 21{,}000 - 150p$.

a) Find a model that expresses the revenue R as a function of the price *p*.
b) What is the domain of R?
c) What unit price should be used to maximize revenue?
d) If this price is charged, what is the maximum revenue?
e) How many units are sold at this price?
f) Graph R.
g) What price should Texas Instruments charge to collect at least \$675,000 in revenue?

Solution: a) The Revenue R is $R(p) = x(p)p$, where $x(p) = 21{,}000 - 150p$.

$$\begin{aligned} R(p) &= x(p)p \\ &= (21{,}000 - 150p)p \\ &= -150p^2 + 21{,}000p \end{aligned}$$

b) Because x represents the number of calculators sold, you have $x \ge 0$, so $21{,}000 - 150p \ge 0$.

Solving this linear inequality

$$\begin{aligned} 21{,}000 - 150p &\ge 0 \\ 21{,}000 &\ge 150p \\ \frac{21{,}000}{150} &\ge p \\ 140 &\ge p \end{aligned}$$

gives $p \le 140$. In addition, Texas Instruments will charge only a positive price for the calculator, so $p > 0$ too. Combining these inequalities gives the domain of R, which is $\{p \mid 0 < p \le 140\}$.

c) The function R is a quadratic function with $a = -150$, $b = 21{,}000$, and $c = 0$. Because $a < 0$, the vertex is the highest point on the parabola. The revenue R is a maximum when the price *p* is

$$p = \frac{-b}{2a}$$

$$p = \frac{-21{,}000}{2(-150)}$$

$$p = \frac{-21{,}000}{-300}$$

$$p = \$70.00$$

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d) The maximum revenue R is

$$\begin{aligned} R(70) &= -150(70)^2 + 21{,}000(70) \\ &= -150(4900) + 1{,}470{,}000 \\ &= -735{,}000 + 1{,}470{,}000 \\ &= \$735{,}000 \end{aligned}$$

e) The number of calculators sold is given by the demand equation $x(p) = 21{,}000 - 150p$,
At a price of $p = \$70$, $x(70) = 21{,}000 - 150(70)$

$$= 10{,}500$$

So, 10,500 calculators are sold.

f) To graph R, plot the intercept $(140, 0)$ and the vertex $(70, 735{,}000)$. See Figure 23 for the graph.

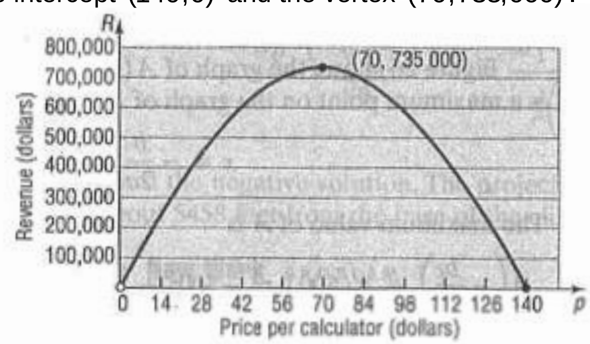

Figure 23

g) Graph $R = 675{,}000$ and $R(p) = -150p^2 + 21{,}000p$ on the same Cartesian Plane. See Figure 24.
You find where the graphs intersect by solving the equation $675{,}000 = -150p^2 + 21{,}000p$.

$675{,}000 = -150p^2 + 21{,}000p$

$150p^2 - 21{,}000p + 675{,}000 = 0$

$p^2 - 140p + 4{,}500 = 0$ divide by 150

$(p - 50)(p - 90) = 0$ factor

$\Rightarrow (p - 50) = 0$ or $(p - 90) = 0$ Zero-Product Property

$p = 50$ or $p = 90$

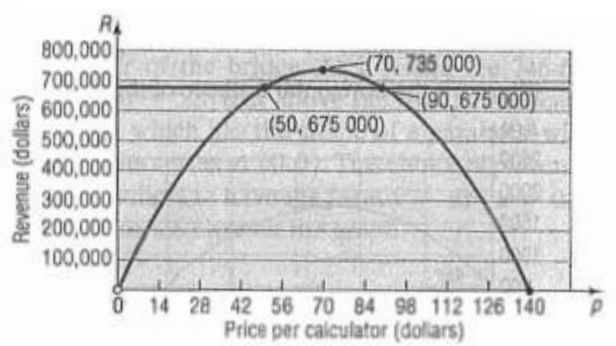

Figure 24

The graphs intersect at $(50, 675{,}000)$ and $(90, 675{,}000)$. Based on the graph in Figure 24, Texas Instruments should charge between \$50 and \$90 to earn at least \$675,000 in revenue.

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Example 4: Maximizing the Area Enclosed by a Fence

A farmer has 2000 yards of fence to enclose a rectangular field. What are the dimensions of the rectangle that encloses the most area?

Solution: Figure 25 illustrates the situation. The available fence represents the perimeter of the rectangle. If x is the length and w is the width, then

$$2x + 2w = P$$
$$2x + 2w = 2000$$

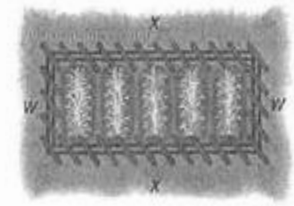

Figure 25

The area A of the triangle is $A = lw$

$$A = xw$$

To express area A in terms of a single variable, solve the equation $2x + 2w = 2000$ for w and substitute the result in $A = xw$.

$$2x + 2w = 2000$$
$$2w = -2x + 2000$$
$$\frac{2w}{2} = \frac{-2x + 2000}{2}$$
$$w = -x + 1000$$

So, the area function is $A(x) = xw$

$$A(x) = x(-x + 1000)$$
$$A(x) = -x^2 + 1000x$$

Because $a < 0$, the vertex is a maximum point on the graph of A(x). The maximum value occurs at

$$x_v = \frac{-b}{2a} = \frac{-1000}{2(-1)} = \frac{-1000}{-2} = 500$$

Figure 26 shows the graph of $A(x) = -x^2 + 1000x$.

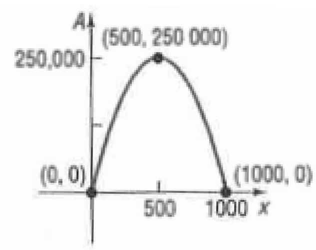

Figure 26 $A(x) = -x^2 + 1000x$

The maximum value of A(x) is $A(x_v) = A(500)$

$$= -(500)^2 + 1000(500)$$
$$= -250{,}000 + 500{,}000$$
$$= 250{,}000$$

So, the largest rectangle that can be enclosed by 2000 yards of fencing has an area of 250,000 square yards. Its dimensions are 500 by 500 yards.

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Example 5: Analyzing the Motion of a Projectile

A projectile is fired from a cliff 500 feet above the water at an inclination of 45° to the horizontal, with a muzzle velocity of 400 feet per second. From physics, the height *h* of the projectile above the water can be modeled by $h(x)=\dfrac{-32x^2}{(400)^2}+x+500$, where *x* is the horizontal distance of the projectile from the base of the cliff. See Figure 27.

a) Find the maximum height of the projectile.

b) How far from the base of the cliff will the projectile strike the water?

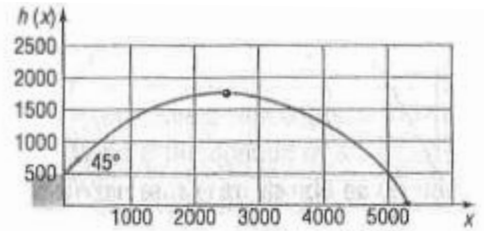

Figure 27

Solution: a) The height of the projectile is given by a quadratic function,

$$h(x)=\frac{-32x^2}{(400)^2}+x+500$$
$$=\frac{-32x^2}{160{,}000}+x+500.$$
$$=\frac{-x^2}{5000}+x+500.$$

You are looking for the maximum value of *h*. Because $a<0$, the maximum value occurs at the vertex, whose x-coordinate is $x_V=\dfrac{-b}{2a}$

$$=\frac{-1}{2\left(\frac{-1}{5000}\right)}$$
$$=\frac{-5000}{-2}$$
$$=2500$$

The maximum height of the projectile is $h(2500)=\dfrac{-(2500)^2}{5000}+2500+500$

$$=\frac{-6{,}250{,}000}{5000}+3000$$
$$=-1250+3000$$
$$=1750 \text{ feet}$$

b) The projectile will strike the water when its height is zero. To find the distance x traveled, solve the equation $h(x)=0$.

$$h(x)=0$$
$$\frac{-x^2}{5000}+x+500=0$$

The discriminant is $b^2-4ac=(1)^2-4\left(\dfrac{-1}{5000}\right)(500)$

$$=1+4(0.1)$$
$$=1+0.4$$
$$=1.4$$

Section 3.4 – Build Quadratic Models from Verbal Descriptions (continued)

Example 5 (continued):

$$\text{So, } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-1 \pm \sqrt{1.4}}{2\left(\frac{-1}{5000}\right)}$$
$$= \frac{-1 \pm \sqrt{1.4}}{\frac{-1}{2500}}$$
$$= \frac{-1 \pm \sqrt{1.4}}{\frac{-1}{2500}}$$
$$= \begin{cases} -458.04 \\ 5458.04 \end{cases}$$
$$\approx \begin{cases} -458 \\ 5458 \end{cases} \quad \text{Discard the negative solution.}$$

The projectile will strike the water at a distance of about 5458 feet from the base of the cliff.

Example 6: The Golden Gate Bridge

The Golden Gate Bridge, a suspension bridge, spans the entrance to San Francisco Bay. Its 746-foot-fall towers are 4200 feet apart. The bridge is suspended from two huge cables more than 3 feet in diameter; the 90-foot-wide roadway is 220 feet above the water. The cables are parabolic in shape and touch the road surface at the center of the bridge. Find the height of the cable above the road at a distance of 1000 feet from the center.

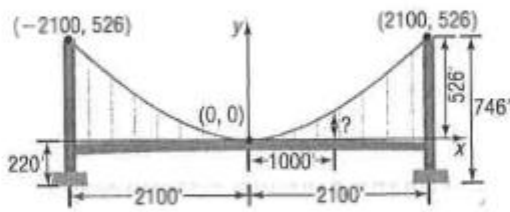

Figure 28

Solution: See Figure 28. Begin by choosing the placement of the coordinate axes so that the x-axis coincides with the road surface and the origin coincides with the center of the bridge. As a result, the 746-foot towers will be vertical (height $746 - 220 = 526$ feet above the road) and located 2100 feet from the center. Also, the cable, which has the shape of a parabola, will extend from the towers, open up, and have its vertex at $(0,0)$. This choice of placement of the axes enables the equation of the parabola to have the form $y = ax^2, a > 0$. Note that the points $(-2100,526)$ and $(2100,526)$ are on the graph.

Use these facts to find the value of *a* in $y = ax^2$.

$$y = ax^2$$
$$526 = a(2100)^2$$
$$a = \frac{526}{(2100)^2}$$

The equation of the parabola is $y = \frac{526}{(2100)^2}x^2$.

When $x = 1000$, the height of the cable is

$$y = \frac{526}{(2100)^2}(1000)^2$$
$$\approx 119.274$$
$$\approx 119.27 \text{ feet}$$

The cable is 119.27 feet above the road at a distance of 1000 feet from the center of the bridge.

All material has been taken from *Precalculus,* by M. Sullivan, 10th Edition