Section 3.5 – Inequalities Involving Quadratic Functions

Solve Inequalities Involving a Quadratic Function

In this section we will solve inequalities that involve quadratic functions. We will accomplish this by using their graphs. For example, to solve the inequality $ax^2+bx+c>0,\ a\neq 0,$ graph the function $f(x)=ax^2+bx+c$ and, from the graph, determine where it is above the x-axis – that is, where $f(x)>0$. To solve the inequality $ax^2+bx+c<0,\ a\neq 0,$ graph the function $f(x)=ax^2+bx+c$ and determine where the graph is below the x-axis – that is, where $f(x)<0$. If the inequality is not strict, include the x-intercepts, if any, in the solution.

Example 1: Solve the inequality $x^2-4x-12\leq 0$ and graph the solution set.

Solution: Graph the function $f(x)=x^2-4x-12$.

y-intercept: set $x=0$

$$f(0)=0^2-4(0)-12$$
$$=0-0-12$$
$$=-12$$

y-int: $(0,-12)$

x-intercepts, if any: set $f(x)=0$

$$x^2-4x-12=0$$
$$(x-6)(x+2)=0$$
$$\Rightarrow (x-6)=0 \text{ or } (x+2)=0$$
$$x=6 \text{ or } x=-2$$

x-int: $(-2,0)$ and $(6,0)$

Vertex: $x_v=\dfrac{-b}{2a}=\dfrac{-(-4)}{2(1)}=\dfrac{4}{2}=2$

$f(x_v)=f(2)=2^2-4(2)-12=4-8-12=-4-12=-16$

Vertex: $(2,-16)$

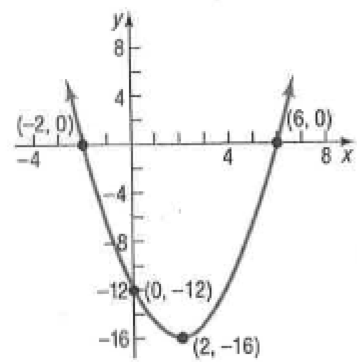


Figure 33 $f(x)=x^2-4x-12$

The graph is below the x-axis for $-2<x<6$. Because the original inequality is not strict, include the x-intercepts. The solution set is $\{x\mid -2\leq x\leq 6\}$ or, using interval notation $[-2,6]$. The graph of the solution set is below.

Example 2: Solve the inequality $2x^2<x+10$ and graph the solution set.

Solution: There are two solution options.

Option 1: Rearrange the inequality so that 0 is on the right side.

$$2x^2<x+10$$
$$2x^2-x-10<0$$

Graph the function $f(x)=2x^2-x-10$ to find where $f(x)<0$.

y-intercept: set $x=0$

$$f(0)=2(0)^2-0-10$$
$$=0-0-10$$
$$=-10$$

y-int: $(0,-10)$

x-intercepts, if any: set $f(x)=0$

$$2x^2-x-10=0$$
$$(2x-5)(x+2)=0$$
$$\Rightarrow (2x-5)=0 \text{ or } (x+2)=0$$
$$2x=5 \text{ or } x=-2$$
$$x=\frac{5}{2}$$

x-int: $\left(\frac{5}{2},0\right)$ and $(-2,0)$

Section 3.5 – Inequalities Involving Quadratic Functions (continued)

Find the vertex:

$$\text{Vertex: } x_v = \frac{-b}{2a} = \frac{-(-1)}{2(2)} = \frac{1}{4}$$

$$f(x_v) = f\left(\frac{1}{4}\right) = 2\left(\frac{1}{4}\right)^2 - \frac{1}{4} - 10 = 2\left(\frac{1}{16}\right) - \frac{1}{4} - 10 = \frac{1}{8} - \frac{2}{8} - 10 = \frac{-1}{8} - 10 = \frac{-81}{8} \text{ or } = -10.125$$

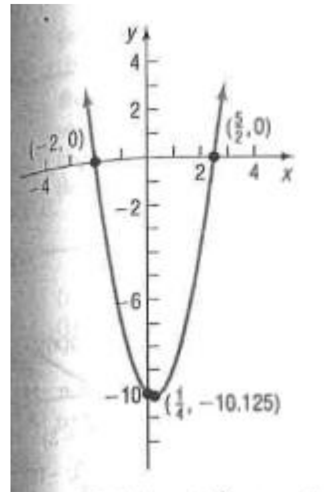


Figure 35 $f(x) = 2x^2 - x - 10$

$$\text{Vertex: } = \left(\frac{1}{4}, \frac{-81}{8}\right) \text{ or } = \left(\frac{1}{4}, -10.125\right)$$

The graph is below the x-axis $(f(x) < 0)$ between $x = -2$ and $x = \frac{5}{2}$. Because the inequality is strict, the solution set is $\left\{x \mid -2 < x < \frac{5}{2}\right\}$ or, using interval notation, $\left(-2, \frac{5}{2}\right)$.

Option 2: If $f(x) = 2x^2$ and $g(x) = x + 10$, then the inequality to be solved is $f(x) < g(x)$. Graph the functions $f(x) = 2x^2$ and $g(x) = x + 10$. See Figure 36. The graphs intersect where $f(x) = g(x)$.

Set $f(x) = g(x)$.

$$2x^2 = x + 10$$
$$2x^2 - x - 10 = 0$$
$$(2x - 5)(x + 2) = 0$$
$$\Rightarrow (2x - 5) = 0 \text{ or } (x + 2) = 0$$
$$2x = 5 \text{ or } x = -2$$
$$x = \frac{5}{2}$$

Now, $g\left(\frac{5}{2}\right) = \frac{5}{2} + 10 = \frac{25}{2}$ and $g(-2) = -2 + 10 = 8$

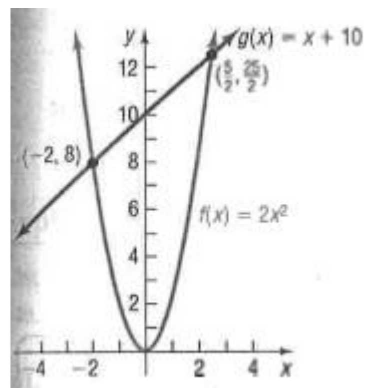


Figure 36

So, the graphs intersect at $(-2, 8)$ and $\left(\frac{5}{2}, \frac{25}{2}\right)$. To solve $f(x) < g(x)$, find where the graph of f is below the graph of g. This happens between the points of intersection. Because the inequality is strict, the solution set is $\left\{x \mid -2 < x < \frac{5}{2}\right\}$ or, using interval notation, $\left(-2, \frac{5}{2}\right)$.

See Figure 37 for the graph of the solution set.

Figure 37

Section 3.5 – Inequalities Involving Quadratic Functions (continued)

Example 3: Solve the inequality $x^2 + x + 1 > 0$ and graph the solution set.

Solution: Graph the function $f(x) = x^2 + x + 1$.

y-intercept: set $x = 0$

$$f(0) = (0)^2 + 0 + 1 = 0 + 0 + 1 = 1$$

y-int: $(0,1)$

x-intercepts, if any: set $f(x) = 0$

$$x^2 + x + 1 = 0$$

$$\text{discriminant} = b^2 - 4ac = 1^2 - 4(1)(1) = 1 - 4 = -3$$

Since the disc < 0, there are no real solutions, so there are no x-intercepts.

Vertex: $x_v = \frac{-b}{2a} = \frac{-(1)}{2(1)} = \frac{-1}{2}$

$$f(x_v) = f\left(\frac{-1}{2}\right) = \left(\frac{-1}{2}\right)^2 + \frac{-1}{2} + 1 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$$

So, the vertex is at $\left(\frac{-1}{2}, \frac{3}{4}\right)$.

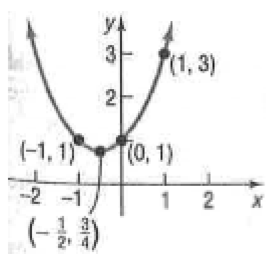
Determine a few more points on the graph:

$$f(1) = (1)^2 + 1 + 1 = 1 + 2 = 3$$

$$f(-1) = (-1)^2 + -1 + 1 = 1 + 0 = 1$$

So, the points $(1,3)$ and $(-1,1)$ are also on the graph. See Figure 38.

Figure 38 $f(x) = x^2 + x + 1$



The graph of f lies above the x-axis for all x. The solution set is the set of all real numbers, or $(-\infty, \infty)$. See Figure 39.

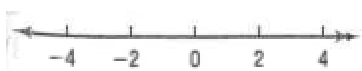


Figure 39