Section 4.1 – Polynomial Functions and Models – Day 2

Know Properties of the Graph of a Polynomial Function

To graph most polynomials of degree 3 or higher requires advanced techniques. However, if you can locate the x-intercepts of the graph, then algebraic techniques can be used to obtain the graph. The x-intercepts divide the x-axis into open intervals and, on each such interval, the graph of the polynomial will be either above or below the x-axis.

Figure 9 shows the graph of a polynomial function with four x-intercepts. Notice that at the x-intercepts, the graph must either cross the x-axis or touch the x-axis. Consequently, between consecutive x-intercepts the graph is either above the x-axis or below the x-axis.

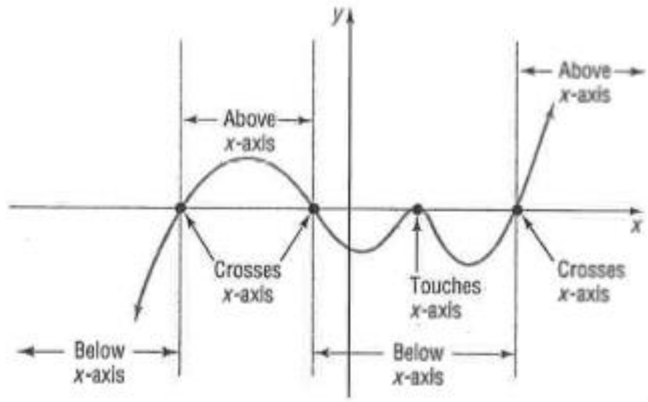


Figure 9 Graph of a polynomial function

If a polynomial function f is factored completely, it is easy to locate the x-intercepts of the graph by solving the equation $f(x)=0$ and using the Zero-Product Property.

Definition: If f is a function and r is a real number for which $f(r)=0$, then r is called a **real zero** of f, or **root** of f.

As a consequence of this definition, the following statements are equivalent.

If r is a real zero of f then:
1) r is a real zero of a polynomial function f.
2) r is an x-intercept of the graph of f.
3) $(x-r)$ is a factor of f.
4) r is a solution to the equation $f(x)=0$.

So, the real zeros of a polynomial function are the x-intercepts of its graph, and they are found by solving the equation $f(x)=0$.

Example 3: Finding a Polynomial Function from Its Zeros

a) Find a polynomial function of degree 3 whose zeros are -4, -3, and 1.
b) Use a graphing utility to graph the polynomial found in part a) to verify your result.

a) If r is a real zero of a polynomial function f, then $(x-r)$ is a factor of f. This means that $(x-(-4))=(x+4)$, $(x-(-3))=(x+3)$, and $(x-1)$ are factors of f. So, any polynomial function of the form $f(x)=a(x+4)(x+3)(x-1)$ where *a* is a nonzero real number, qualifies. The value of *a* causes a vertical stretch, vertical compression, or a reflection, but it does not affect the x-intercepts of the graph.

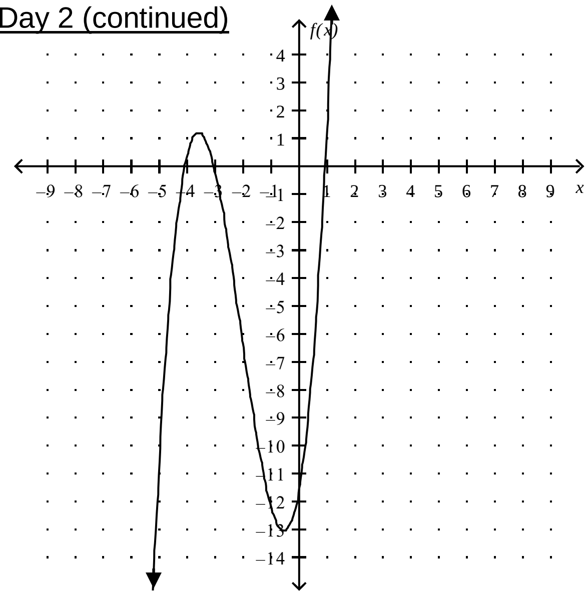
b) Choose to graph f with $a=1$. Then
$$\begin{aligned} f(x) &= (x+4)(x+3)(x-1) \\ &= (x^2+7x+12)(x-1) \\ &= x^3+6x^2+5x-12 \end{aligned}$$

Section 4.1 – Polynomial Functions and Models – Day 2 (continued)

Example 3: (continued)

Graph $f(x) = x^3 + 6x^2 + 5x - 12$

Notice that the x-intercepts are at $(0,-4)$, $(0,-3)$, and $(0,1)$.



If the same factor $(x - r)$ occurs more than once, then r is called a **repeated**, or **multiple**, **zero** of f. This leads to the following definition.

Definition: If $(x - r)^m$ is a factor of a polynomial f and $(x - r)^{m+1}$ is not a factor of f, then r is called a **zero of multiplicity m of f**. (Also known as a multiple root or root of multiplicity m.)

So, the multiplicity of a zero is the number of times its corresponding factor occurs.

Example 4: Identifying Zeros and Their Multiplicities

Find the zeros, and their multiplicities, of the polynomial function $f(x) = 4(x+3)(x-5)^2\left(x+\frac{2}{3}\right)^6$.

Degree of function $f(x)$: $f(x) = 4(x)(x^2)(x^6) + \ldots$

$= 4x^9 + \ldots$, so the degree $= 9$

Find the zeros: set $f(x) = 0$.

$$4(x+3)(x-5)^2\left(x+\frac{2}{3}\right)^6 = 0$$

$$\Rightarrow \quad (x+3) = 0 \text{ or } (x-5)^2 = 0 \text{ or } \left(x+\frac{2}{3}\right)^6 = 0$$

$$\Rightarrow \quad (x+3) = 0 \text{ or } (x-5) = 0 \text{ or } \left(x+\frac{2}{3}\right) = 0$$

$$\Rightarrow \quad x = -3 \text{ or } x = 5 \text{ or } x = \frac{-2}{3}$$

So,

-3 is a zero of multiplicity 1 because the exponent on the factor $(x+3)$ is 1.

5 is a zero of multiplicity 2 because the exponent on the factor $(x-5)$ is 2.

$\frac{-2}{3}$ is a zero of multiplicity 6 because the exponent on the factor $\left(x+\frac{2}{3}\right)$ is 6.

If you add the multiplicities, you get the degree of the polynomial. ($1+2+6=9$, the degree)

Section 4.1 – Polynomial Functions and Models – Day 2 (continued)

Suppose that it is possible to completely factor a polynomial function and, as a result, locate all the x-intercepts of its graph (the real zeros of the function). These x-intercepts then divide the x-axis into open intervals and, on each such interval, the graph of the polynomial will be either above or below the x-axis over the entire interval.

Example 5: Graphing a Polynomial using its x-intercepts

Consider the following polynomial: $f(x)=(x+1)^2(x-2)$

a) Find the x- and y-intercepts of the graph of f.
b) Use the x-intercepts to find the intervals on which the graph of f is above the x-axis and the intervals on which the graph of f is below the x-axis.
c) Locate other points on the graph, and connect all the points plotted with a smooth, continuous curve.

Solution: a) For the y-intercept, set $x=0$. So, $f(0)=(0+1)^2(0-2)$

$$=(1)^2(-2)$$
$$=(1)(-2)$$
$$=-2$$

So, the y-intercept is $(0,-2)$.

For the x-intercepts, set $f(x)=0$. So, $(x+1)^2(x-2)=0$

$$\Rightarrow (x+1)^2=0 \quad \text{or} \quad (x-2)=0$$
$$(x+1)=0 \quad \text{or} \quad x=2$$
$$x=-1$$

So, the x-intercepts are $(-1,0)$ and $(2,0)$.

b) The two x-intercepts divide the x-axis into three intervals:

$(-\infty,-1)$ $(-1,2)$ $(2,\infty)$

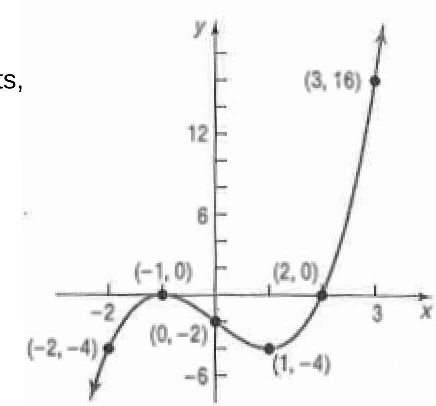
Since the graph of f crosses or touches the x-axis only at $x=-1$ and $x=2$, it follows that the graph of f is either above the x-axis $[f(x)>0]$ or below the x-axis $[f(x)<0]$ on each of these three intervals. To see where the graph lies, pick a number in each interval, evaluate f there, and see whether the value is positive (above the x-axis) or negative (below the x-axis).

	−1	2	x
Interval	$(-\infty,-1)$	$(-1,2)$	$(2,\infty)$
Number chosen	−2	1	3
Value of *f*	$f(-2)=-4$	$f(1)=-4$	$f(3)=16$
Location of graph	Below *x*-axis	Below *x*-axis	Above *x*-axis
Point on graph	(−2, −4)	(1, −4)	(3, 16)

c) In constructing the table, you obtain three additional points on the graph:

$(-2,-4)$, $(1,-4)$, and $(3,16)$

The figure illustrates these points, the intercepts, and a smooth, continuous curve.



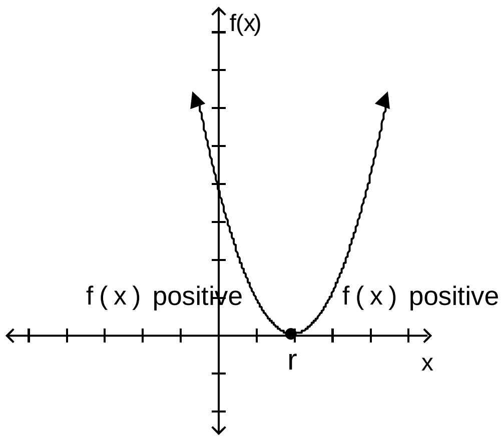
$f(x)=(x+1)^2(x-2)$

Section 4.1 – Polynomial Functions and Models – Day 2 (continued)

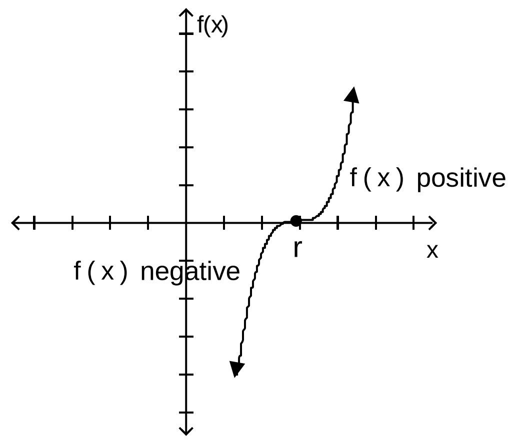
Look again at the table in Example 5. Since the graph of $f(x)=(x+1)^2(x-2)$ is below the x-axis on both sides of -1, the graph of f touches the x-axis at $x=-1$, a zero of multiplicity 2. Since the graph of f is below the x-axis for $x<2$ and above the x-axis for $x>2$, the graph of f crosses the x-axis at $x=2$, a zero of multiplicity 1.

This suggests the following:

If r is a zero of even multiplicity then:
The graph of $f(x)$ touches the x-axis at r.
The sign of $f(x)$ does not change from one side to the other side of r.



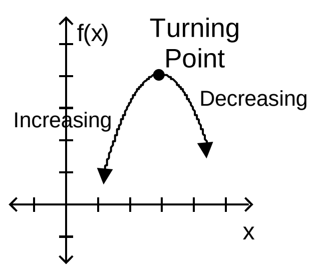
If r is a zero of odd multiplicity then:
The graph of $f(x)$ crosses the x-axis at r.
The sign of $f(x)$ changes from one side to the other side of r.



Look again at the graph in Example 5. You cannot be sure just how low the graph actually goes between $x=-1$ and $x=2$. But you do know that somewhere in the interval $(-1,2)$ the graph of f must change direction (from decreasing to increasing).

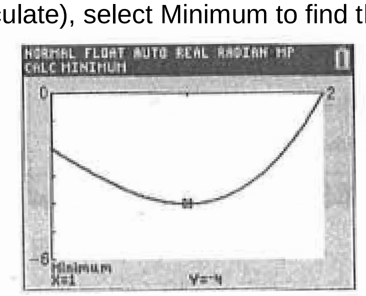
The points at which a graph changes direction (decreasing to increasing or increasing to decreasing) are called **turning points**. Each turning point yields either a local maximum or a local minimum (Sec 2.3).

$f(x)=ax(x-3)^2(x-1)$



Theorem: Turning Points
If f is a polynomial function of degree n, then f has at most $(n-1)$ turning points.
If the graph of a polynomial function f has $n-1$ turning points, then the degree of f is at least n.

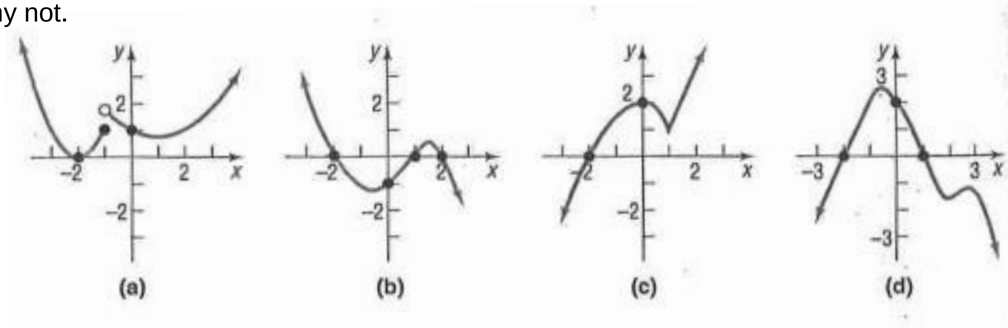
A graphing calculator can be used to locate the turning points of a graph.
Graph $Y_1=(x+1)^2(x-2)$. Use 2nd Trace (for Calculate), select Minimum to find the location of the turning point for $0<x<2$. The minimum is at $(1,-4)$.



Section 4.1 – Polynomial Functions and Models – Day 2 (continued)

Example 6: Identifying the Graph of a Polynomial Function

Which of the graphs in the figure below could be the graph of a polynomial function? For those that could, list the real zeros and state the least degree the polynomial can have. For those that could not, say why not.



a) The graph cannot be the graph of a polynomial function because of the gap that occurs at $x=-1$. Remember, the graph of a polynomial function is continuous – no holes or gaps.

b) The graph could be the graph of a polynomial function because the graph is smooth and continuous. It has three real zeros: $-2, 1,$ and 2. Since the graph has two turning points, the degree of the polynomial function must be at least 3.

c) The graph cannot be the graph of a polynomial function because of the cusp at $x=1$. Remember, the graph of a polynomial function is smooth.

d) The graph could be the graph of a polynomial function. It has two real zeros: -2 and 1. Since the graph has three turning points, the degree of the polynomial function is at least 4.

End Behavior

Look again at the graph in Example 5. For very large values of x, either positive or negative, the graph of $f(x)=(x+1)^2(x-2)$ looks like the graph of $y=x^3$. To see why, write f in the following form:

$$
\begin{aligned}
f(x) &= (x+1)^2(x-2) \\
&= (x^2+2x+1)(x-2) \\
&= x^3+2x^2+x-2x^2-4x-2 \\
&= x^3-3x-2 \\
&= x^3\left(1-\frac{3}{x^2}-\frac{2}{x^3}\right)
\end{aligned}
$$

Now, for large values of x, either positive or negative, the terms $\frac{3}{x^2}$ and $\frac{2}{x^3}$ are close to 0, so for large values of x,

$$
\begin{aligned}
f(x) &= (x+1)^2(x-2) \\
&= x^3\left(1-\frac{3}{x^2}-\frac{2}{x^3}\right) \\
&\approx x^3
\end{aligned}
$$

The behavior of the graph of a function for large values of x, either positive or negative, is referred to as its **end behavior.**

End Behavior Theorem:

For large values of x, either positive or negative, the graph of the polynomial $f(x)=a_n x^n + a_{n-1}x^{n-1}+\ldots+a_1x+a_0$ resembles the graph of the power function $y=a_n x^n$.

Section 4.1 – Polynomial Functions and Models – Day 2 (continued)

For example, if $f(x) = -2x^3 + 5x^2 + x - 4$, then the graph of f will behave like the graph of $y = -2x^3$ for very large values of x, either positive or negative. You can see that the graphs of f and $y = -2x^3$ "behave" the same by considering Table 4 and Figure 14.

Table 4

x	$f(x)$	$y = -2x^3$
10	−1,494	−2,000
100	−1,949,904	−2,000,000
500	−248,749,504	−250,000,000
1,000	−1,994,999,004	−2,000,000,000

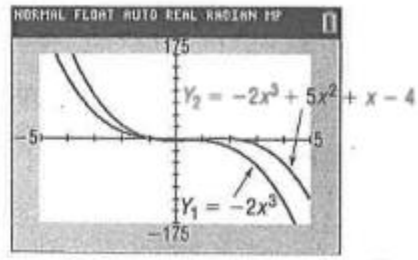


Figure 14

Notice that as x becomes a larger and larger positive number, the values of f become larger and larger negative numbers. When this happens, we say that f is unbounded in the negative direction. Rather than using words to describe the behavior of the graph of the function, we explain its behavior using notation. We can symbolize "the value of f becomes a larger and larger negative number as x becomes a larger and larger positive number" by writing $f(x) \to -\infty$ as $x \to \infty$ (read "the values of f approach negative infinity as x approaches infinity"). In calculus, limits are used to convey these ideas. There we use the symbolism $\lim_{x \to \infty} f(x) = -\infty$, read "the limit of $f(x)$ as x approaches infinity equals negative infinity," to mean that $f(x) \to -\infty$ as $x \to \infty$.

Remember that infinity (∞) and negative infinity $(-\infty)$ are not numbers. Rather, they are symbols that represent unboundedness.

When we say that the value of a limit equals infinity (or negative infinity), we mean that the values of the function are unbounded in the positive (or negative) direction and call the limit an **infinite limit**. When we discuss limits as x becomes unbounded in the negative direction or unbounded in the positive direction, we are discussing **limits at infinity**.

Based on the preceding theorem and the previous discussion on power functions, the end behavior of a polynomial function can be of only four types. See Figure 15 below.

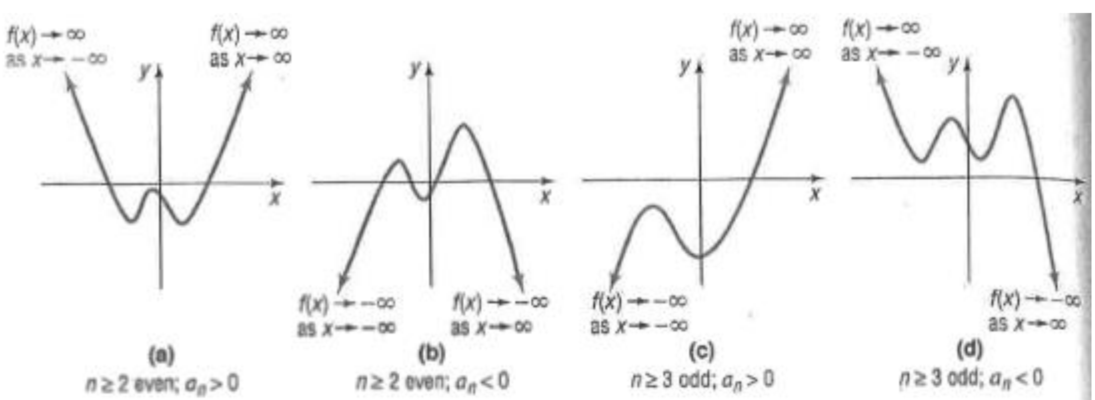


Figure 15 End behavior of $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$

For example, if $f(x) = -2x^4 + x^3 + 4x^2 - 7x + 1$, the graph of f will resemble the graph of the power function $y = -2x^4$ for large $|x|$. The graph of f will behave like Figure 15(b) for large $|x|$.

Section 4.1 – Polynomial Functions and Models – Day 2 (continued)

Example 7: Identifying the Graph of a Polynomial Function

Which of the graphs in Figure 16 could be the graph of $f(x) = x^4 + ax^3 + bx^2 - 5x - 6$ where $a > 0, b > 0$?

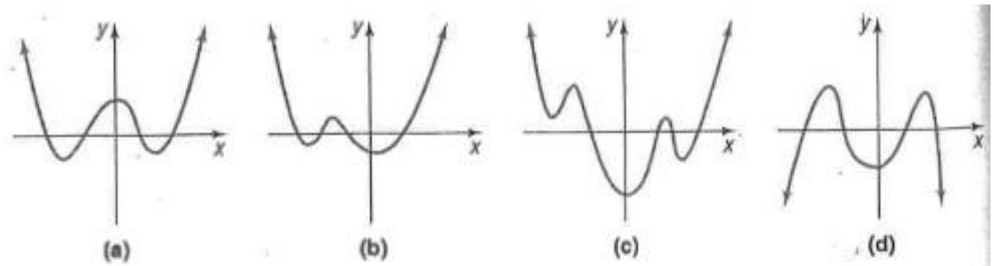
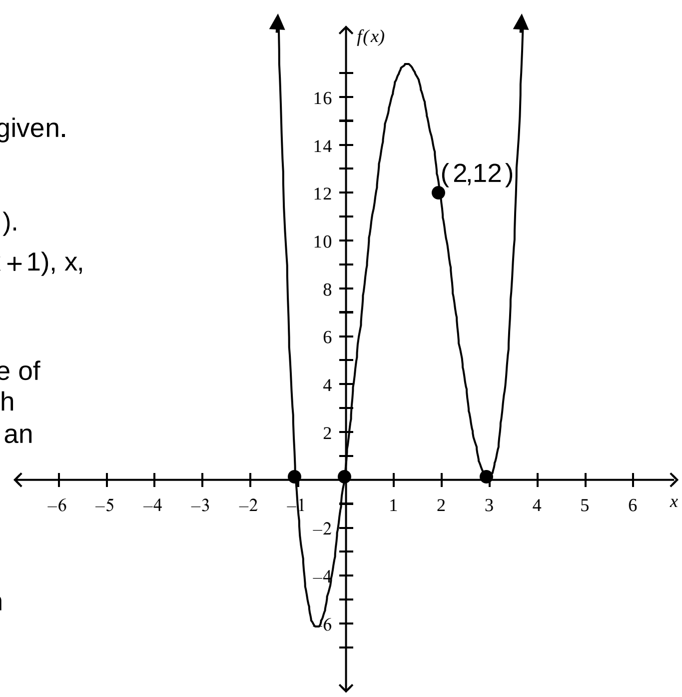


Figure 16

Solution: The y-intercept of f is $f(0) = -6$. So, we can eliminate the graph in Figure 16(a), whose y-intercept is positive.
We are not able to solve $f(x) = 0$ to find the x-intercepts of f, so we move on to investigate the turning points of each graph. Since f is of degree 4, the graph of f has at most $4 - 1 = 3$ turning points. We can eliminate the graph in Figure 16(c) because that graph has 5 turning points.
Now we look at the end behavior. For large values of x, the graph of f will behave like the graph of $y = x^4$. This eliminates the graph in Figure 16(d), whose end behavior is like the graph of $y = -x^4$.

So, only the graph in Figure 16(b) could be the graph of $f(x) = -x^4 + ax^3 + bx^2 - 5x - 6$ where $a > 0, b > 0$.

Example 8: Write a Polynomial Function from its Graph



Write a polynomial function whose graph is given.
Use the smallest degree possible.

The x-intercepts are $(-1, 0), (0, 0)$, and $(3, 0)$.
So, the polynomial must have the factors $(x+1)$, x, and $(x-3)$, respectively.

There are three turning points, so the degree of the polynomial must be at least 4. The graph touches the x-axis at $x = 3$, so 3 must have an even multiplicity. The graph crosses the x-axis at $x = -1$ and $x = 0$, so -1 and 0 must have odd multiplicities.

Using the smallest degree possible, you can write $f(x) = ax(x-3)^2(x+1)$.

The point $(2, 12)$ lies on the graph, so use it to find the coefficient *a*.

$(2,12)$: $12 = a(2)(2-3)^2(2+1)$

$12 = 2a(-1)^2(3)$

$12 = 6a(1)$

$12 = 6a$

$a = 2$

The polynomial function $f(x) = 2x(x-3)^2(x+1)$ will have the graph shown above.

Section 4.1 – Polynomial Functions and Models – Day 2 (continued)

Summary: Graph of a Polynomial Function $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0,\ a_n \neq 0.$

Degree of the polynomial function $f(x)$: n
y-intercept: $f(0) = a_0$
The graph is smooth and continuous
Maximum number of turning points: $n-1$
At a zero of even multiplicity: The graph of $f(x)$ touches the x-axis
At a zero of odd multiplicity: The graph of $f(x)$ crosses the x-axis
Between zeros, the graph of $f(x)$ is either above the x-axis or below the x-axis.
End behavior: For large $|x|$, the graph of $f(x)$ behaves like the graph of $y = a_n x^n$.

Analyze the Graph of a Polynomial Function

Example 9: Analyze the factored form of the polynomial function $f(x) = (2x+1)(x-3)^2$.
Expand the polynomial:

Step 1: Determine the end behavior of the graph of the function.

$$
\begin{aligned}
f(x) &= (2x+1)(x-3)^2 \\
&= (2x+1)(x^2-6x+9) \\
&= 2x^3 - 12x^2 + 18x + x^2 - 6x + 9 \\
&= 2x^3 - 11x^2 + 12x + 9
\end{aligned}
$$

The polynomial function f is of degree 3. The graph of f behaves like $y = 2x^3$ for large values of $|x|$.

Step 2: Find the x- and y-intercepts of the graph of the function.

The y-intercept is $f(0) = 9$. To find the x-intercepts, solve $f(x) = 0$.

$$
\begin{aligned}
&f(x) = 0 \\
&(2x+1)(x-3)^2 = 0 \\
&\Rightarrow (2x+1) = 0 \text{ or } (x-3)^2 = 0 \\
&2x = -1 \text{ or } x - 3 = 0 \\
&x = \frac{-1}{2} \text{ or } x = 3
\end{aligned}
$$

The x-intercepts are $\left(\frac{-1}{2}, 0\right)$ and $(3,0)$.

Step 3: Determine the zeros of the function and their multiplicity. Use this information to determine whether the graph crosses or touches the x-axis at each x-intercept.

The zeros of f are $\frac{-1}{2}$ and 3. The zero $\frac{-1}{2}$ is a zero of multiplicity 1, so the graph of f crosses the x-axis at $x = \frac{-1}{2}$. The zero 3 is a zero of multiplicity 2, so the graph of f touches the x-axis at $x = 3$.

Step 4: Determine the maximum number of turning points on the graph of the function.

Because the polynomial function is of degree 3 (Step 1), the graph of the function will have at most $3 - 1 = 2$ turning points.

Section 4.1 – Polynomial Functions and Models – Day 2 (continued)

Step 5: Put all the information from Steps 1 through 4 together to obtain the graph of f. To help establish the y-axis scale, find additional points on the graph on each side of any x-intercept.

Figure 18(a) illustrates the information obtained from Steps 1 through 4. Evaluate f at $-1, 1,$ and 4 to help establish the scale on the y-axis.

You find that $f(-1)=-16$, $f(1)=12$, and $f(4)=9$, so plot the points $(-1,-16)$, $(1,12)$, and $(4,9)$.

The graph of f is given in Figure 18(b).

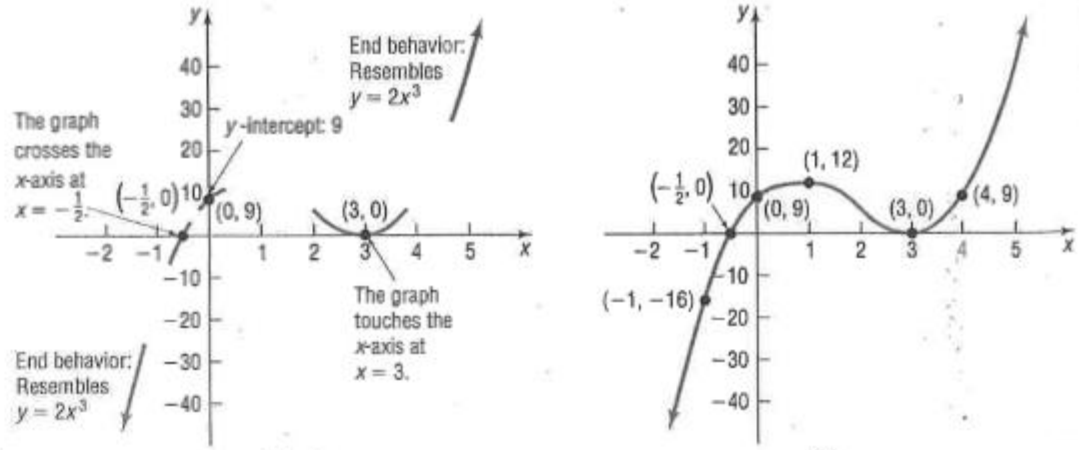


Figure 18 (a) (b)

Summary: Analyzing the Graph of a Polynomial Function

Step 1: Determine the end behavior of the graph of the function.

Step 2: Find the x- and y-intercepts of the graph of the function.

Step 3: Determine the zeros of the function and their multiplicity. Use this information to determine whether the graph crosses or touches the x-axis at each x-intercept.

Step 4: Determine the maximum number of turning points on the graph of the function.

Step 5: Use the information in Steps 1 through 4 to draw a complete graph of the function. To help establish the y-axis scale, find additional points on the graph on each side of any x-intercept.

Example 10: Graph the polynomial function $f(x)=x^2(x+4)$. Give all specifics.

$$f(x)=x^2(x+4)$$
$$=x^3+4x^2$$

$f(x)$ is of degree n = 3

End behavior: For large values of x, $f(x)$ resembles $p(x)=x^3$.

Find the x- and y-intercepts.

x-intercept: Set $f(x)=0$.

$x^2(x+4)=0$

$\Rightarrow\ x^2=0$ or $(x+4)=0$

$\Rightarrow\ x=0$ or $x=-4$

x-intercepts: $(0,0)$ and $(-4,0)$.

y-intercept: Set $x=0$.

$f(0)=0^2(0+4)$

$=0(4)$

$=0$

y-intercept: $(0,0)$

-4 is a zero of multiplicity 1: odd $\Rightarrow$ crosses at -4

0 is a zero of multiplicity 2: even $\Rightarrow$ touches at 0

(If you add the multiplicities, you get the degree of the polynomial (degree n = 3).)

Section 4.1 – Polynomial Functions and Models – Day 2 (continued)

Example 10 continued:

Maximum number of turning points:

Degree n = 3

Max number of turning points $= n-1$

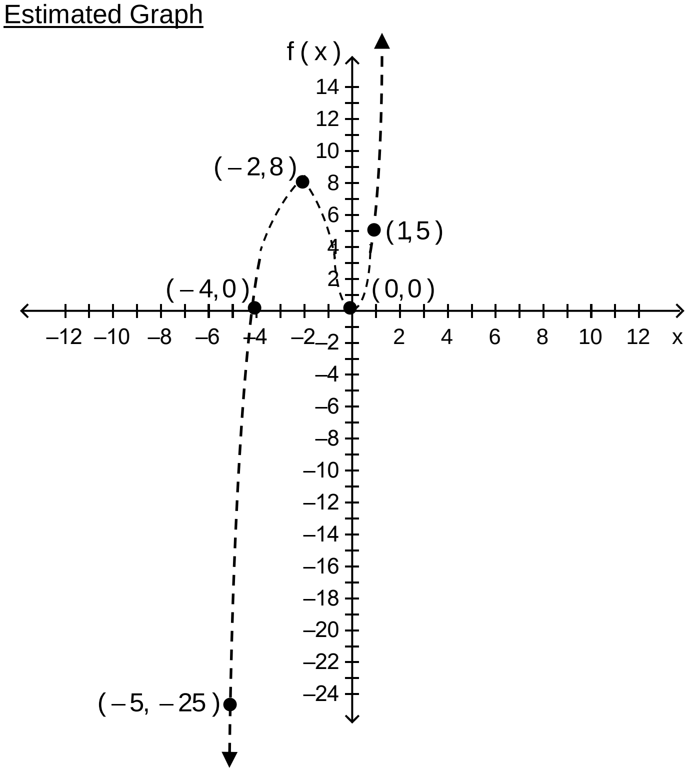
$= 3-1$

$= 2$

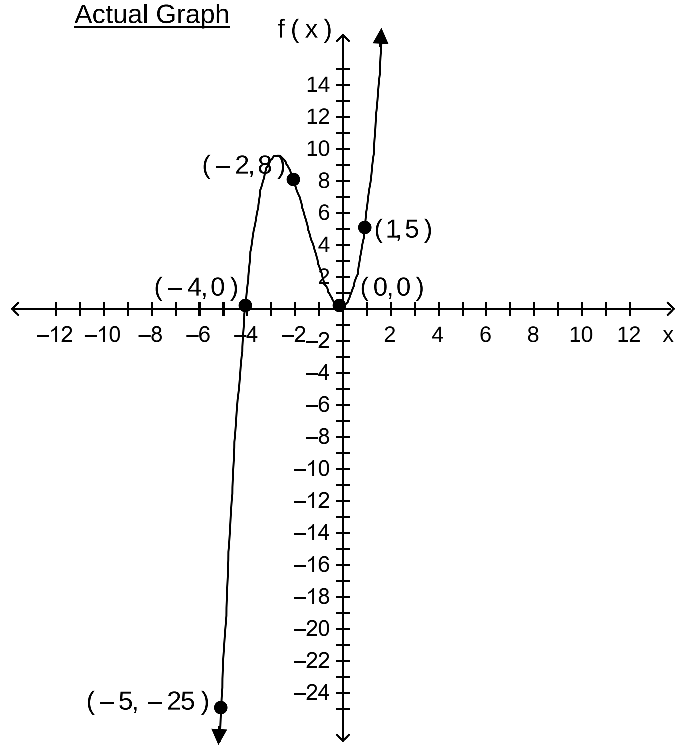
The x-intercepts divide the x-axis into three intervals:

	$-\infty < x < -4$	$-4 < x < 0$	$0 < x < \infty$
x^2	+	+	+
$(x+4)$	−	+	+
$f(x) = x^2(x+4)$	−	+	+
$\Rightarrow$ The graph is	Below the x-axis	Above the x-axis	Above the x-axis
Test point:	-5	-2	1
f (test point)	$f(-5) = (-5)^2(-5+4)$ $= 25(-1)$ $= -25$	$f(-2) = (-2)^2(-2+4)$ $= 4(2)$ $= 8$	$f(1) = (1)^2(1+4)$ $= 1(5)$ $= 5$
Points on graph:	$(-5, -25)$	$(-2, 8)$	$(1, 5)$

Estimated Graph



Actual Graph



All material has been taken from *Precalculus,* by M. Sullivan, 10th Edition