Section 4.2 – Properties of Rational Functions – Day 2

Finding Asymptotes:

Finding Vertical Asymptotes:

A rational function $R(x) = \dfrac{p(x)}{q(x)}$, in lowest terms, will have a vertical asymptote $x = r$, if $x - r$ is a factor of the denominator q(x).

To find the vertical asymptote, ask yourself, "what value of x will make the denominator equal to zero?," i.e., what are the zeros of the denominator? If $R(x)$ is in lowest terms and r is a zero of the denominator of $R(x)$, then $R(x)$ has a vertical asymptote at x = r.

Example 3: Find the vertical asymptotes, if any, of the graph of each rational function.

a) $R(x) = \dfrac{x}{x^2 - 9}$

$= \dfrac{x}{(x+3)(x-3)}$ lowest terms

Set $(x+3)(x-3) = 0$

$\Rightarrow x + 3 = 0$ or $x - 3 = 0$

$x = -3$ or $x = 3$

$\Rightarrow$ VA: $x = 3, x = -3$

b) $h(x) = \dfrac{2x^2}{x^2 + 3}$ lowest terms

Set $x^2 + 3 = 0$

But, $x^2 + 3 > 0 \ \forall x$

The denominator has no real zeros, so there are no vertical asymptotes.

c) $f(x) = \dfrac{x^2 - 4}{x^2 + 2x - 8}$

$= \dfrac{(x+2)\cancel{(x-2)}}{(x+4)\cancel{(x-2)}}$

$= \dfrac{(x+2)}{(x+4)}$ lowest terms

Set $x + 4 = 0$

$x = -4$

$\Rightarrow$ VA: $x = -4$

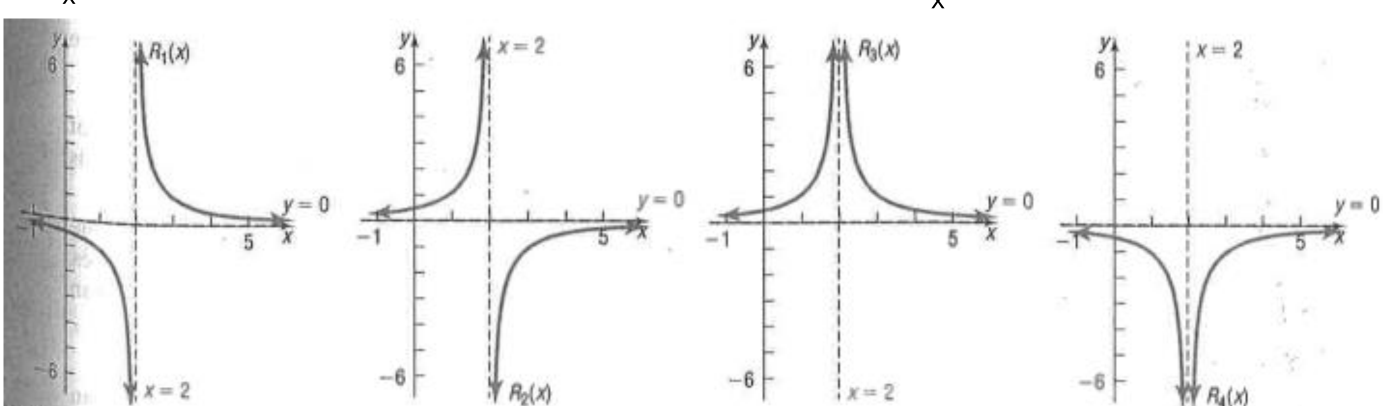
So, from the examples, you see that rational functions can have no vertical asymptotes, one vertical asymptote, or more than one vertical asymptote. The graph of a function will never intersect any of its vertical asymptotes.

Recall from Figure 15 in Section 4.1 (Ch4Sec1Day2 notes, page 6) that the end behavior of a polynomial function is always one of four types. For polynomials of odd degree, the ends of the graph go in opposite directions (one up and one down), whereas for polynomials of even degree, the ends go in the same direction (both up or both down).

For a rational function in lowest terms, the multiplicities of the zeros in the denominator can be used in a similar fashion to determine the behavior of the graph around each vertical asymptote. Consider the following four functions, each with a single vertical asymptote, x = 2.

$R_1(x) = \dfrac{1}{x-2}$ $\quad R_2(x) = \dfrac{-1}{x-2}$ $\quad R_3(x) = \dfrac{1}{(x-2)^2}$ $\quad R_4(x) = \dfrac{-1}{(x-2)^2}$

Figure 29 shows the graphs of each function. The graphs of R_1 and R_2 are transformations of the graph of $y = \dfrac{1}{x}$, and the graphs of R_3 and R_4 are transformations of the graph of $y = \dfrac{1}{x^2}$.



(a) Odd multiplicity $\lim_{x \to 2^-} R_1(x) = -\infty$ $\lim_{x \to 2^+} R_1(x) = \infty$

(b) Odd multiplicity $\lim_{x \to 2^-} R_2(x) = \infty$ $\lim_{x \to 2^+} R_2(x) = -\infty$

(c) Even multiplicity $\lim_{x \to 2^-} R_3(x) = \infty$ $\lim_{x \to 2^+} R_3(x) = \infty$

(d) Even multiplicity $\lim_{x \to 2^-} R_4(x) = -\infty$ $\lim_{x \to 2^+} R_4(x) = -\infty$

Figure 29

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Based on Figure 29, you can make the following conclusions:

- If the multiplicity of the zero that gives rise to a vertical asymptote is odd, the graph approaches ∞ on one side of the vertical asymptote and approaches $-\infty$ on the other side.
- If the multiplicity of the zero that gives rise to the vertical asymptote is even, the graph approaches either ∞ or $-\infty$ on both sides of the vertical asymptote.

These results are true in general and will be helpful when graphing rational functions in the next section.

Finding Horizontal and Oblique Asymptotes:

To find horizontal or oblique asymptotes, you need to know how the value of the function behaves as $x \to -\infty$ or $x \to \infty$. That is, you need to determine the end behavior of the function. This can be done by examining the degrees of the numerator and denominator, and the respective power functions that each resembles. For example, consider the rational function $f(x) = \dfrac{x+3}{6x^2+2x+1}$. The degree of the numerator, 1, is less than the degree of the denominator, 2. When $|x|$ is very large, the numerator of R can be approximated by the power function $y = x$, and the denominator can be approximated by the power function $y = 6x^2$. This means

$$f(x) = \frac{x+3}{6x^2+2x+1} \approx \frac{x}{6x^2} \text{ for } |x| \text{ very large}$$

$$= \frac{1}{6x} \to 0 \text{ as } x \to -\infty \text{ or } x \to \infty$$

which shows that the line $y = 0$ is a horizontal asymptote. This result is true for all rational functions that are <u>**proper**</u> (that is, the degree of the numerator is less than the degree of the denominator). If a rational function is <u>**improper**</u> (that is, if the degree of the numerator is greater than or equal to the degree of the denominator), there could be a horizontal asymptote, an oblique asymptote, or neither. The following summary details how to find horizontal or oblique asymptotes.

Finding a Horizontal or Oblique Asymptote of a Rational Function R

Consider the rational function

$$R(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0}$$

in which the degree of the numerator is n and the degree of the denominator is m.

1) If $n < m$ (the degree of the numerator is less than the degree of the denominator), then R is a proper rational function, and the line $y = 0$ is a horizontal asymptote.

2) If $n = m$ (the degree of the numerator equals the degree of the denominator), the line $y = \dfrac{a_n}{b_m}$ is a horizontal asymptote. (That is, the horizontal asymptote equals the ratio of the leading coefficients.)

3) If $n = m+1$ (the degree of the numerator is one more than the degree of the denominator), the line $y = ax + b$ is an oblique asymptote, which is the quotient found using long division.

4) If $n \geq m+2$ (the degree of the numerator is two or more than the degree of the denominator), there are no horizontal or oblique asymptotes. The end behavior of the graph will resemble the power function $y = \dfrac{a_n}{b_m}x^{n-m}$.

Note: A rational function will never have both a horizontal asymptote and an oblique asymptote. Also, a rational function may have neither a horizontal nor an oblique asymptote.

Example 4: Find the horizontal asymptote of $f(x)=\dfrac{x+3}{6x^2+2x+1}$.

$f(x)$ is proper: degree numerator = 1, degree denominator = 2

So, degree numerator < degree denominator

So, $f(x)$ is proper $\Rightarrow$ HA: $y=0$

Why? What happens to $f(x)$ as $x\to\infty$? $f(x)\approx\dfrac{x}{6x^2}$ as x gets large

$$=\frac{1}{6x}$$

And $\dfrac{1}{6x}\to 0$ as $x\to\infty$ or $x\to-\infty$.

If a rational function is improper (deg num $\ge$ deg denom), use long division to find the asymptote.

Example 5: a) $f(x)=\dfrac{2x^4-x^2}{x^3+x^2+2}$

Improper: deg num = 4
deg denom = 3
deg num $\ge$ deg denom

$$\begin{array}{r|l} & 2x-2 \\ \hline x^3+x^2+2 & 2x^4+0x^3-x^2+0x+0 \\ & -(2x^4+2x^3+4x) \\ \hline & -2x^3-x^2-4x+0 \\ & -(-2x^3-2x^2-4) \\ \hline & x^2-4x+4 \end{array}$$

So, $f(x)=\dfrac{2x^4-x^2}{x^3+x^2+2}$

$$=2x-2+\frac{x^2-4x+4}{x^3+x^2+2}$$

As $x\to\pm\infty$, $f(x)\to 2x-2$. $\Rightarrow$ oblique asymptote OA: $y=2x-2$

b) $g(x)=\dfrac{6x^2+x+4}{2x^2-1}$

Improper: deg num = 2, deg denom = 2
deg num $\ge$ deg denom

So, $g(x)=\dfrac{6x^2+x+4}{2x^2-1}$

$$=3+\frac{x+7}{2x^2-1}$$

$$\begin{array}{r|l} & 3 \\ \hline 2x^2-1 & 6x^2+x+4 \\ & -(6x^2-3) \\ \hline & x+7 \end{array}$$

As $x\to\pm\infty$, $g(x)\to 3$. $\Rightarrow$ horizontal asymptote HA: $y=3$

c) $h(x)=\dfrac{3x^5+2x^3+7}{x^3+1}$

Improper: deg num = 5, deg denom = 3
deg num $\ge$ deg denom

So, $h(x)=\dfrac{3x^5+2x^3+7}{x^3+1}$

$$=3x^2+2+\frac{-3x^2+5}{x^3+1}$$

$$\begin{array}{r|l} & 3x^2+2 \\ \hline x^3+1 & 3x^5+0x^4+2x^3+0x^2+0x+7 \\ & -(3x^5+3x^2) \\ \hline & 2x^3-3x^2+0x+7 \\ & -(2x^3+2) \\ \hline & -3x^2+5 \end{array}$$

As $x\to\pm\infty$, $h(x)\to 3x^2+2$, a quadratic
$\Rightarrow$ $h(x)$ has no horizontal or oblique asymptotes.