## Section 4.3 – The Graph of a Rational Function

### Analyzing the Graph of a Rational Function $R(x)$

You will analyze the graph of a rational function $R(x)=\dfrac{p(x)}{q(x)}$ by applying a seven (7) step process. These seven steps give you a lot of information about the graph, such as position and shape, before you actually graph it.

Step 1: Factor the numerator and denominator of $R(x)$. Find the domain of the rational function.

Step 2: Write $R(x)$ in lowest terms.

Step 3: Find and plot the intercepts of the graph.
Use multiplicity to determine the behavior of the graph of $R(x)$ at each x-intercept.
The x-intercepts satisfy $R(x)=0$ The y-intercept is $R(0)$.

$$\frac{p(x)}{q(x)}=0$$
$$\Rightarrow\ p(x)=0$$

Step 4: Find the vertical asymptotes. Graph each vertical asymptote using a dashed line.
Determine the behavior of the graph of $R(x)$ on either side of each vertical asymptote.

Step 5: Find the horizontal or oblique asymptote, if one exists.
Find points, if any, at which the graph of $R(x)$ intersects this asymptote.
Graph the asymptote using a dashed line.
Plot any points at which the graph of $R(x)$ intersects the asymptote.

Step 6: Use the zeros of the numerator and denominator of $R(x)$ to divide the x-axis into intervals.
Determine where the graph of $R(x)$ is above or below the x-axis by choosing a number in each interval and evaluating $R(x)$ there.
Plot the points found.

Step 7: Use the results obtained in Steps 1 through 6 to graph $R(x)$.

When graphing rational functions on the calculator, using dot mode (versus connected mode) will eliminate the extraneous vertical lines that are not part of the graph.

Keep in mind that in Step 2, you write $R(x)$ in lowest terms. Remember that zeros of the denominator of a rational function give rise to either vertical asymptotes or holes in the graph. Holes result when you cancel like factors in the process of writing $R(x)$ in lowest terms, given that a similar factor does not still remain in the denominator.

Example 1: Analyze the graph of the rational function $R(x)=\dfrac{4x^2-4x}{x^2+4x-5}$.

Step 1: Factor the numerator and denominator of $R(x)$. Find the domain of the rational function.

$$R(x)=\frac{4x^2-4x}{x^2+4x-5}=\frac{4x(x-1)}{(x+5)(x-1)}$$

Set the denominator equal to 0: $(x+5)(x-1)=0$

$$\Rightarrow\ x+5=0 \text{ or } x-1=0$$
$$x=-5 \text{ or } x=1$$

The domain of $R(x)$ is $\{x\,|\,x\neq-5, x\neq1\}$.

Step 2: Write $R(x)$ in lowest terms.

$$R(x) = \frac{4x^2 - 4x}{x^2 + 4x - 5}$$
$$= \frac{4x(x-1)}{(x+5)(x-1)}$$
$$= \frac{4x}{(x+5)}$$

In simplifying R(x), we cancelled factors of $(x-1)$, so there is a hole at $x = 1$.

Step 3: Find and plot the intercepts of the graph.
Use multiplicity to determine the behavior of the graph of $R(x)$ at each x-intercept.

x-intercept: set $R(x) = 0$

$$\frac{4x}{(x+5)} = 0$$
$$\Rightarrow 4x = 0$$
$$x = 0$$
$$\Rightarrow \text{x-int:} (0,\ 0)$$

The multiplicity of 0, 1, is odd, so the graph will cross the x-axis at $x = 0$.

y-intercept: set $x = 0$

$$R(0) = \frac{4(0)}{0+5}$$
$$= \frac{0}{5}$$
$$= 0$$
$$\Rightarrow \text{y-int:} (0,\ 0)$$

Step 4: Find the vertical asymptotes. Graph each vertical asymptote using a dashed line.
Determine the behavior of the graph of $R(x)$ on either side of each vertical asymptote.

Set the denominator equal to zero: $x + 5 = 0$

$$x = -5$$
$$\Rightarrow \text{VA: } x = -5$$

The multiplicity of $x = -5$ is 1, which is odd. Therefore, the graph will approach $\infty$ on one side of the vertical asymptote, and will approach $-\infty$ on the other side.

Step 5: Find the horizontal or oblique asymptote, if one exists.
Find points, if any, at which the graph of $R(x)$ intersects this asymptote.
Graph the asymptote using a dashed line.
Plot any points at which the graph of $R(x)$ intersects the asymptote.

The degree of the numerator is 1, and the degree of the denominator is 1. So, the degree of the numerator is greater than or equal to the degree of the denominator $\Rightarrow R(x)$ is improper.

$$\begin{array}{r} 4 \\ x+5 \overline{)\ 4x+0} \\ -(4x+20) \\ \hline -20 \end{array}$$

So, $R(x) = 4 + \dfrac{-20}{x+5}$

As $x \to \pm\infty$, $R(x) \to 4$.

$\Rightarrow$ HA: $y = 4$

Look for intersections of the function $R(x)$ with this horizontal asymptote:

Solve $R(x) = 4$.

$$\frac{4x}{(x+5)} = 4$$
$$4x = 4(x+5)$$
$$4x = 4x + 20$$

$0 = 20$ A false statement. $\Rightarrow$ The graph of $R(x)$ does not intersect the line $y = 4$.

Step 6: Use the zeros of the numerator and denominator of $R(x)$ to divide the x-axis into intervals. Determine where the graph of $R(x)$ is above or below the x-axis by choosing a number in each interval and evaluating $R(x)$ there.
Plot the points found.

Using the original function $R(x)$: Zeros of the numerator: $0, 1$
Zeros of the denominator: $-5, 1$

$\Rightarrow$ Use $-5$, $0$, and 1 to divide the number line into four intervals.

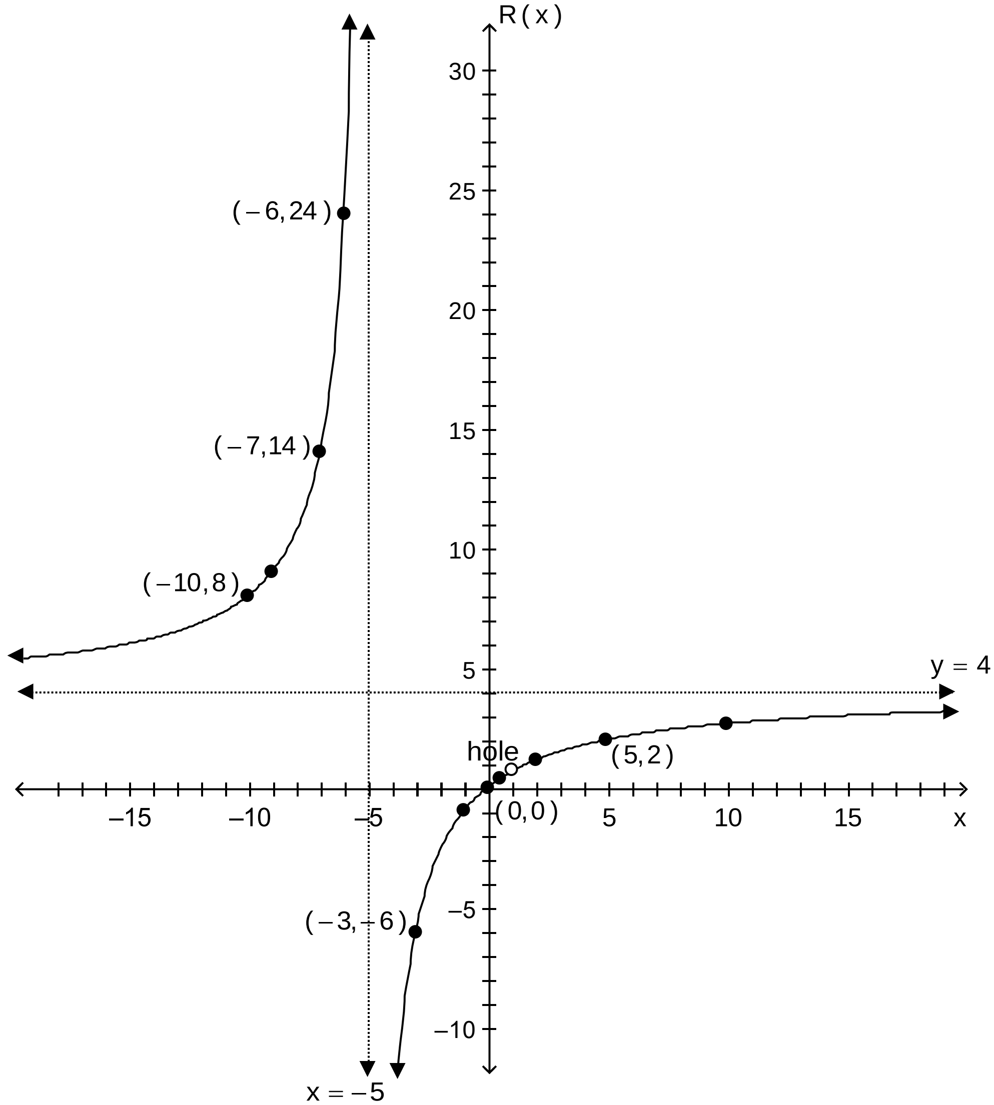
| | $-\infty < x < -5$ | $-5 < x < 0$ | $0 < x < 1$ | $1 < x < \infty$ |
|---|---|---|---|---|
| Test Value: | $-6$ | $-1$ | $\frac{1}{2}$ | $2$ |
| R (Test Value): | $R(-6) = \frac{168}{7}$ | $R(-1) = \frac{8}{-8}$ | $R\left(\frac{1}{2}\right) = \frac{-1}{\frac{-11}{4}}$ | $R(2) = \frac{16-8}{4+8-5}$ |
| $R(x) = \frac{4x^2 - 4x}{x^2 + 4x - 5}$ | $= 24$ | $= -1$ | $= \frac{4}{11}$ $\approx 0.36$ | $= \frac{8}{7}$ $\approx 1.14$ |
| $\Rightarrow$ The graph is | Above the x-axis | Below the x-axis | Above the x-axis | Above the x-axis |
| Points on graph: | $(-6, 24)$ | $(-1, -1)$ | $(0.5, 0.36)$ | $(2, 1.14)$ |
| Extra points: | $(-7, 14)$ | $(-2, -2.67)$ | $(0.1, 0.08)$ | $(3, 1.5)$ |
| | $(-9, 9)$ | $(-3, -6)$ | $(0.7, 0.49)$ | $(5, 2)$ |
| | $(-10, 8)$ | $(-4, -16)$ | $(0.9, 0.61)$ | $(10, 2.67)$ |

From Step 2, you know you have a hole at $x = 1$. Find the coordinates of the hole by evaluating $R(x)$ *in lowest terms* at $x = 1$.

$$R(x) = \frac{4x}{(x+5)}, \text{ so } R(1) = \frac{4(1)}{(1+5)} = \frac{4}{6} = \frac{2}{3}$$

Therefore, the hole is at $\left(1, \frac{2}{3}\right)$.

As this example shows, **the zeros of the denominator of a rational function give rise to either vertical asymptotes or holes on the graph.**




All material has been taken from ***Precalculus,*** by M. Sullivan, 10th Edition