Section 4.4 – Polynomial and Rational Inequalities

Solving a Quadratic Inequality

In this section, you will solve inequalities that involve polynomials of degree 2 and higher, along with inequalities that involve rational functions. The approach follows the same methodology that we used to solve inequalities involving quadratic functions in Section 3.5.

To solve these polynomial inequalities rearrange them so that the polynomial is on the left side and 0 is on the right side. Factor the polynomial, use the zeros to divide the real number line into intervals, choose test numbers in each interval, and evaluate each factor to see if it is positive or negative.

Example 1: Solve $x^2 - x - 20 > 0$

Old method:

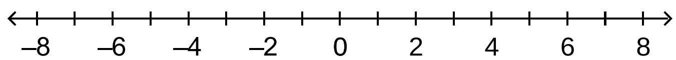
$(x+4)(x-5) > 0 \Rightarrow$ Both factors are positive or negative

$x+4>0$ and $x-5>0$ OR $x+4<0$ and $x-5<0$

$x>-4$ and $x>5$ $\quad$ $x<-4$ and $x<5$

=> $x>5$ $\quad$ => $x<-4$ $\quad$ => The solution is $\{x \mid x<-4 \text{ or } x>5\}$

Graph:



OR: New method:

$(x+4)(x-5) > 0$

Separate the number line into three intervals based on the zeros

Zeros at:

$(x+4)(x-5) = 0$

=> $x+4=0$ or $x-5=0$

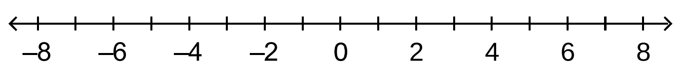
$x=-4$ or $x=5$

	$-\infty < x < -4$	$-4 < x < 5$	$5 < x < \infty$
$x+4$	−	+	+
$x-5$	−	−	+
$(x+4)(x-5)$	+	−	+

You want $x^2 - x - 20 > 0$, so circle the columns with the positive (+) sign in the bottom row.

=> The solution is $\{x \mid x<-4 \text{ or } x>5\}$.

Graph:



Example 2: Solve $x^2 \le 3x + 10$

$x^2 - 3x - 10 \le 0$

$(x-5)(x+2) \le 0$

Separate the number line into three intervals based on the zeros

Zeros at:

$(x-5)(x+2) = 0$

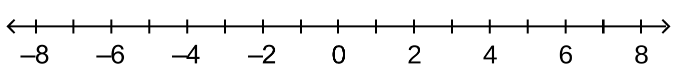
=> $x-5=0$ or $x+2=0$

$x = 5$ or $x=-2$

	$-\infty < x < -2$	$-2 < x < 5$	$5 < x < \infty$
$x+2$	−	+	+
$x-5$	−	−	+
$(x-5)(x+2)$	+	−	+

You want $x^2 - 3x - 10 \le 0$, so circle the column with the negative $(-)$ sign in the bottom row.
You can have equality, so include the endpoints. => The solution is $\{x \mid -2 \le x \le 5\}$.

Graph:



But what if the equation has no real solution?

Theorem: If p is a polynomial and the polynomial equation $p(x) = 0$ has no real solutions, the polynomial is either always positive or always negative.

Example 3: $x^2 + x + 2 = 0$ has no real solutions (discriminant $b^2 - 4ac = -7$).

It is a parabola that opens upward (since the coefficient on the x^2 term is positive).
Test a point, say, x = 0: $p(0) = 2$, a positive number

=> By the theorem, $x^2 + x + 2 > 0$ for all x.

Section 4.4 – Polynomial and Rational Inequalities (continued)

Example 4: Solve the inequality $x^5 \le x^2$.

$$x^5 - x^2 \le 0$$
$$x^2(x^3 - 1) \le 0$$
$$x^2(x-1)(x^2+x+1) \le 0$$

Recall: $a^3 - b^3 = (a-b)(a^2+ab+b^2)$

Zeros at: $x^2(x-1)(x^2+x+1) = 0$

$\Rightarrow$ $x^2 = 0$ or $x - 1 = 0$ or $x^2 + x + 1 = 0$

$x = 0$ or $x = 1$ discriminant:

$b^2 - 4ac = 1^2 - 4(1)(1)$ disc < 0
$= 1 - 4$ $\Rightarrow$ no real solution
$= -3$

use the Theorem (on Pg 1):

test x = 0:
$p(0) = 0^2 + 0 + 1$
$= 0 + 1$
$= 1$
> 0

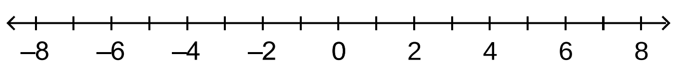
So, by the theorem, $x^2 + x + 1$ is always positive (no zero).

Separate the number line into three intervals based on the zeros:

	$-\infty < x < 0$	$0 < x < 1$	$1 < x < \infty$
x^2	+	+	+
$x - 1$	−	−	+
$x^2 + x + 1$	+	+	+
$x^2(x-1)(x^2+x+1)$	−	−	+

You want $x^2(x-1)(x^2+x+1) \le 0$, so circle the columns with a negative $(-)$ sign in the bottom row. You can have equality, so include the endpoints. Thus, for $x^5 \le x^2$, the solution is $\{x \mid x \le 1\}$ in set notation or $(-\infty, 1]$ in interval notation.

Graph:



Follow a similar process for solving a rational inequality. The sign of a rational expression depends on the sign of its numerator and the sign of its denominator.

Example 5: Solve $\dfrac{(x+2)(4-x)}{(x-1)^2} > 0$. Domain: $\{x \mid x \ne 1\}$

Numerator Zeros at: $(x+2)(4-x) = 0$ or Denominator Zeros at: $(x-1)^2 = 0$

$\Rightarrow$ $x + 2 = 0$ or $4 - x = 0$ or $x - 1 = 0$

$x = -2$ or $x = 4$ or $x = 1$

Separate the real number line into intervals using the zeros of the numerator and the denominator

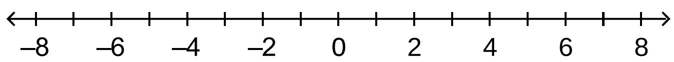
	$-\infty < x < -2$	$-2 < x < 1$	$1 < x < 4$	$4 < x < \infty$
$x + 2$	−	+	+	+
$4 - x$	+	+	+	−
$(x-1)^2$	+	+	+	+
$\dfrac{(x+2)(4-x)}{(x-1)^2}$	−	+	+	−

You want $\dfrac{(x+2)(4-x)}{(x-1)^2} > 0$, so circle the columns with a positive (+) sign in the bottom row.

Example 5: (continued)

Thus, $\dfrac{(x+2)(4-x)}{(x-1)^2} > 0$ has the solution $\{x \mid -2 < x < 4, \quad x \neq 1\}$.

Graph:



Steps for Solving Polynomial and Rational Inequalities:

1) Write the inequality so that a polynomial or rational expression $f(x)$ is on the left side and zero is on the right side in one of the following forms:

   $f(x) > 0 \qquad f(x) \geq 0 \qquad f(x) < 0 \qquad f(x) \leq 0$

   For rational expressions, be sure that the left side is written as a single quotient, and find the domain of f.

2) Determine the numbers at which the expression $f(x)$ on the left side equals zero and, if the expression is rational, the numbers at which the expression $f(x)$ on the left side is undefined.

3) Use the numbers found in step 2 to separate the real number line into intervals.

4) Select a number C in each interval and evaluate $f(x)$ at the number.
   a) If the value of $f(C)$ is positive, then $f(x) > 0$ for all numbers x in the interval.
   b) If the value of $f(C)$ is negative, then $f(x) < 0$ for all numbers x in the interval.
   If the inequality is not strict $(\geq$ or $\leq)$, include the solutions of $f(x) = 0$ that are in the domain of f in the solution set. Be careful to exclude values of x where f is undefined.

The following are Example 2 and Example 4 from the textbook.

EXAMPLE 2 **How to Solve a Polynomial Inequality Algebraically**

Solve the inequality $x^4 > x$ algebraically, and graph the solution set.

$0 < x < 1$, $f(x)$ will be negative. Similarly, we know that $f(x)$ will be positive for $x > 1$, since the multiplicity of the zero 1 is odd. Therefore, the solution set of $x^4 > x$ is $\{x \mid x < 0 \text{ or } x > 1\}$ or, using interval notation, $(-\infty, 0) \cup (1, \infty)$.

This inequality is equivalent to the one we wish to solve.

Step 2: Determine the real zeros (x-intercepts of the graph) of f.

Find the real zeros of $f(x) = x^4 - x$ by solving $x^4 - x = 0$.

$$x^4 - x = 0$$
$$x(x^3 - 1) = 0 \quad \text{Factor out } x.$$
$$x(x-1)(x^2 + x + 1) = 0 \quad \text{Factor the difference of two cubes.}$$
$$x = 0 \text{ or } x - 1 = 0 \text{ or } x^2 + x + 1 = 0 \quad \text{Set each factor equal to zero and solve.}$$
$$x = 0 \text{ or } x = 1$$

The equation $x^2 + x + 1 = 0$ has no real solutions. Do you see why?

Step 3: Use the zeros found in Step 2 to divide the real number line into intervals.

Use the real zeros to separate the real number line into three intervals:

$$(-\infty, 0) \qquad (0, 1) \qquad (1, \infty)$$

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Step 4: Select a number in each interval, evaluate f at the number, and determine whether $f(x)$ is positive or negative. If $f(x)$ is positive, all values of f in the interval are positive. If $f(x)$ is negative, all values of f in the interval are negative.

Select a test number in each interval found in Step 3 and evaluate $f(x) = x^4 - x$ at each number to determine whether $f(x)$ is positive or negative. See Table 14.

Table 14

	0	1	x
Interval	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
Number chosen	-1	$\frac{1}{2}$	2
Value of f	$f(-1) = 2$	$f\left(\frac{1}{2}\right) = -\frac{7}{16}$	$f(2) = 14$
Conclusion	Positive	Negative	Positive

Since we want to know where $f(x)$ is positive, conclude that $f(x) > 0$ for all numbers x for which $x < 0$ or $x > 1$. Because the original inequality is strict, numbers x that satisfy the equation $x^4 = x$ are not solutions. The solution set of the inequality $x^4 > x$ is $\{x \mid x < 0 \text{ or } x > 1\}$ or, using interval notation, $(-\infty, 0) \cup (1, \infty)$.

Figure 42 shows the graph of the solution set.

NOTE If the inequality is not strict (that is, if it is $\leq$ or $\geq$), include the solutions of $f(x) = 0$ in the solution set. ■

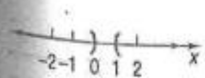

Figure 42

The Role of Multiplicity in Solving Polynomial Inequalities

In Example 2, we used the number -1 and found that $f(x)$ is positive for all $x < 0$. Because the "cut point" of 0 is the result of a zero of odd multiplicity (x is a factor to the first power), we know that the sign of $f(x)$ will change on either side of 0, so for $0 < x < 1$, $f(x)$ will be negative. Similarly, we know that $f(x)$ will be positive for $x > 1$, since the multiplicity of the zero 1 is odd. Therefore, the solution set of $x^4 > x$ is $\{x \mid x < 0 \text{ or } x > 1\}$ or, using interval notation, $(-\infty, 0) \cup (1, \infty)$.

EXAMPLE 4 **How to Solve a Rational Inequality Algebraically**

Solve the inequality $\dfrac{3x^2 + 13x + 9}{(x+2)^2} \leq 3$ algebraically, and graph the solution set.

Step-by-Step Solution

Step 1: Write the inequality so that a rational expression f is on the left side and zero is on the right side.

Rearrange the inequality so that 0 is on the right side.

$$\frac{3x^2 + 13x + 9}{(x+2)^2} \leq 3$$

$$\frac{3x^2 + 13x + 9}{x^2 + 4x + 4} - 3 \leq 0 \quad \text{Subtract 3 from both sides of the inequality; Expand } (x+2)^2$$

$$\frac{3x^2 + 13x + 9}{x^2 + 4x + 4} - 3 \cdot \frac{x^2 + 4x + 4}{x^2 + 4x + 4} \leq 0 \quad \text{Multiply 3 by } \frac{x^2 + 4x + 4}{x^2 + 4x + 4}$$

$$\frac{3x^2 + 13x + 9 - 3x^2 - 12x - 12}{x^2 + 4x + 4} \leq 0 \quad \text{Write as a single quotient.}$$

$$\frac{x - 3}{(x+2)^2} \leq 0 \quad \text{Combine like terms.}$$

Section 4.4 – Polynomial and Rational Inequalities (continued)

Step 2: Determine the real zeros (x-intercepts of the graph) of f and the real numbers for which f is undefined.

The zero of $f(x) = \dfrac{x-3}{(x+2)^2}$ is 3. Also, f is undefined for $x = -2$.

Step 3: Use the zeros and undefined values found in Step 2 to divide the real number line into intervals.

Use the zero and the undefined value to separate the real number line into three intervals:

$$(-\infty, -2) \quad (-2, 3) \quad (3, \infty)$$

Step 4: Select a number in each interval, evaluate f at the number, and determine whether $f(x)$ is positive or negative. If $f(x)$ is positive, all values of f in the interval are positive. If $f(x)$ is negative, all values of f in the interval are negative.

Select a test number in each interval from Step 3, and evaluate f at each number to determine whether $f(x)$ is positive or negative. See Table 15.

Table 15

	-2	3	x
Interval	$(-\infty, -2)$	$(-2, 3)$	$(3, \infty)$
Number chosen	-3	0	4
Value of f	$f(-3) = -6$	$f(0) = -\frac{3}{4}$	$f(4) = \frac{1}{36}$
Conclusion	Negative	Negative	Positive

NOTE If the inequality is not strict ($\le$ or $\ge$), include the solutions of $f(x) = 0$ in the solution set. ■

Since we want to know where $f(x)$ is negative or zero, we conclude that $f(x) \le 0$ for all numbers for which $x < -2$ or $-2 < x \le 3$. Notice that we do not include -2 in the solution because -2 is not in the domain of f. The solution set of the inequality $\dfrac{3x^2 + 13x + 9}{(x+2)^2} \le 3$ is $\{x \mid x < -2 \text{ or } -2 < x \le 3\}$ or, using interval notation, $(-\infty, -2) \cup (-2, 3]$. Figure 44 shows the graph of the solution set. ●

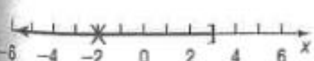

Figure 44

The Role of Multiplicity in Solving Rational Inequalities

In Example 4, we used the number -3 and found that $f(x)$ is negative for all $x < -2$. Because the "cut point" of -2 is the result of a zero of even multiplicity, we know the sign of $f(x)$ will not change on either side of -2, so for $-2 < x < 3$, $f(x)$ will also be negative. Because the "cut point" of 3 is the result of a zero of odd multiplicity, the sign of $f(x)$ will change on either side of 3, so for $x > 3$, $f(x)$ will be positive. Therefore, the solution set of $\dfrac{3x^2 + 13x + 9}{(x+2)^2} \le 3$ is $\{x \mid x < -2 \text{ or } -2 < x \le 3\}$ or, using interval notation, $(-\infty, -2) \cup (-2, 3]$.