Section 4.5 – The Real Zeros of a Polynomial Function – Day 1

In Section 4.1, you were able to identify the real zeros of a polynomial functions because either the polynomial was in factored form or it could be easily factored. But how do you find the real zeros of a polynomial function if it is not factored or cannot easily be factored?

Recall that if r is a real zero of a polynomial function f, then $f(r)=0$, r is an x-intercept of the graph of f, $x-r$ is a factor of f, and r is a solution of the equation $f(x)=0$. For polynomial functions, you have seen the importance of the real zeros for graphing. In most cases, the real zeros of a polynomial function are difficult to find using algebraic methods. No nice formulas like the quadratic formula are available to help you find zeros for polynomials of degree 3 or higher.

Use the Remainder and Factor Theorems

When one polynomial (the dividend) is divided by another (the divisor), a quotient polynomial and a remainder are obtained, where the remainder is the zero polynomial or a polynomial whose degree is less than the degree of the divisor. To check, verify that

(Quotient) (Divisor) + Remainder = Dividend

Theorem: Division Algorithm for Polynomials

If $f(x)$ and $g(x)$ denote polynomial functions and if $g(x)$ is a polynomial whose degree is greater than zero, then there are unique polynomial functions $q(x)$ and $r(x)$ such that

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)} \quad \text{or} \quad \underbrace{f(x)}_{\text{dividend}} = \underbrace{q(x)}_{\text{quotient}}\,\underbrace{g(x)}_{\text{divisor}} + \underbrace{r(x)}_{\text{remainder}}$$

where $r(x)$ is either the zero polynomial or a polynomial of degree less than that of $g(x)$.

In the theorem, $f(x)$ is the dividend, $g(x)$ is the divisor, $q(x)$ is the quotient, and $r(x)$ is the remainder.

Remainder Theorem – Let f be a polynomial function. If $f(x)$ is divided by $x-c$, then the remainder is $f(c)$.

So, the Remainder Theorem tells you that the value of a polynomial function at c equals the remainder when the polynomial is divided by $x-c$.

Example 1: Find the remainder when $f(x)=x^3-2x^2-7x-12$ is divided by $x-4$.

Either long division or synthetic division could be used. For example, use synthetic division to divide $f(x)$ by $x-4$:

$$\begin{array}{r|rrrr} 4 & 1 & -2 & -7 & -12 \\ & & 4 & 8 & 4 \\ \hline & 1 & 2 & 1 & -8 \end{array} \Rightarrow \text{The quotient } q(x)=x^2+2x+1 \text{ and the remainder } R=-8$$

OR

It may be easier to use the Remainder Theorem, which says the remainder is $f(4)$.

$$\begin{aligned} f(4) &= 4^3-2(4)^2-7(4)-12 \\ &= 64-2(16)-28-12 \\ &= 64-32-28-12 \\ &= 32-28-12 \\ &= 4-12 \\ &= -8 \end{aligned}$$

So, this example shows two ways of finding the value of a polynomial function at a number c.

Section 4.5 – The Real Zeros of a Polynomial Function – Day 1 (continued)

An important and useful consequence of the Remainder Theorem is the Factor Theorem.

Factor Theorem – Let f be a polynomial function. Then $x-c$ is a factor of $f(x)$ if and only if $f(c)=0$.

The Factor Theorem actually consists of two separate statements:

1) If $f(c)=0$, then $x-c$ is a factor of $f(x)$.
2) If $x-c$ is a factor of $f(x)$, then $f(c)=0$.

One use of the Factor Theorem is to determine whether a polynomial has a particular factor.

Example 2: Use the Factor Theorem to determine whether the function $f(x)=x^3-5x^2+2x+8$ has the factors $x+1$ and $x+4$.

The Factor Theorem states that if $f(c)=0$, then $x-c$ is a factor of $f(x)$.

a) $x+1=x-(-1)$ Find
$$\begin{aligned} f(-1) &= (-1)^3-5(-1)^2+2(-1)+8 \\ &= -1-5(1)-2+8 \\ &= -1-5-2+8 \\ &= -6+6 \\ &= 0 \end{aligned}$$

So, since $f(-1)=0$, by the Factor Theorem, $x-(-1)=x+1$ is a factor of $f(x)$.

b) To test the factor $x+4$, first write it in the form $x-c$. Since $x+4=x-(-4)$, find $f(-4)$.

$$\begin{array}{r|rrrr} -4 & 1 & -5 & 2 & 8 \\ & & -4 & 36 & -152 \\ \hline & 1 & -9 & 38 & -144 \end{array}$$

By the Remainder Theorem,
$\Rightarrow f(-4)=-144$

So, since $f(-4)\neq 0$, by the Factor Theorem, $x-(-4)=x+4$ is not a factor of $f(x)$.

The next theorem concerns the number of real zeros that a polynomial function may have. In counting the zeros of a polynomial, count each zero as many times as its multiplicity.

Number of Real Zeros Theorem: – A polynomial function cannot have more real zeros than its degree.

Use Descartes' Rule of Signs to Determine the Number of Positive and the Number of Negative Real Zeros of a Polynomial Function

Descartes' Rule of Signs theorem provides information about the number and location of the real zeros of a polynomial function written in standard form (omitting terms with a 0 coefficient). It utilizes the number of variations in the sign of the coefficients of $f(x)$ and $f(-x)$.

For example, the following polynomial has two variations in the signs of the coefficients.

$$f(x)=-4x^5+5x^4+6x^2-5x-8$$

– to +   + to –

Replacing x by $-x$ gives

$$\begin{aligned} f(-x) &= -4(-x)^5+5(-x)^4+6(-x)^2-5(-x)-8 \\ &= 4x^5+5x^4+6x^2+5x-8 \end{aligned}$$

+ to –

which has one variation in sign.

Section 4.5 – The Real Zeros of a Polynomial Function – Day 1 (continued)

Descartes' Rule of Signs Theorem:

Let f denote a polynomial function written in standard form (descending powers of x).

The number of positive real zeros of f either equals the number of variations in the sign of the nonzero coefficients of $f(x)$ or else equals that number less an even integer.

The number of negative real zeros of f either equals the number of variations in the sign of the nonzero coefficients of $f(-x)$ or else equals that number less an even integer.

Descartes' Rule of Signs Theorem gives you information about the number and location of the real zeros of a polynomial function. The theorem assumes the polynomial is written in descending powers of x. You need to count the number of variations in sign of the coefficients of $f(x)$ and $f(-x)$.

Example 3: Determine the number of variations in sign.

a) $f(x) = 3x^2 + 2x - 3$

\+ + ∨ −

$\Rightarrow$ 1 variation in sign

b) $g(x) = 5x^6 - 4x^4 + 6x^2 + 3x - 2$

\+ ∨ − ∨ + + ∨ −

$\Rightarrow$ 3 variations in sign

Example 4: Discuss the real zeros of $f(x) = 2x^7 + 3x^6 - x^5 + 5x^3 + 2x^2 - 4x + 7$.

$f(x)$ is a polynomial of degree 7. $\Rightarrow$ There are at most 7 real zeros. (By the Number of Real Zeros Thm)

$f(x) = 2x^7 + 3x^6 - x^5 + 5x^3 + 2x^2 - 4x + 7$.

\+ + ∨ − ∨ + + ∨ − ∨ +

$\Rightarrow$ 4 variations in sign

So, by Descartes' Rule of Signs there are either 4, 2, or 0 positive real zeros.

$f(-x) = 2(-x)^7 + 3(-x)^6 - (-x)^5 + 5(-x)^3 + 2(-x)^2 - 4(-x) + 7$

$= -2x^7 + 3x^6 + x^5 - 5x^3 + 2x^2 + 4x + 7$

− ∨ + + ∨ − ∨ + + +

$\Rightarrow$ 3 variations in sign

So, by Descartes' Rule of Signs there are either 3 or 1 negative real zeros.

Thus, the graph of f has either 4, 2, or no positive x-intercepts and 3 or 1 negative x-intercepts.

Use the Rational Zeros Theorem to List the Potential Rational Zeros of a Polynomial Function

You now know something about the number of real zeros and how many might be positive or negative.
The following theorem gives you information about the rational zeros of a polynomial with **integer coefficients**.

Rational Zeros Theorem – Let f be a polynomial function of degree 1 or higher of the form

$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \quad a_n \neq 0, a_0 \neq 0$ where each coefficient is an integer.

If $\frac{p}{q}$, in lowest terms, is a rational zero of f, then p must be a factor of a_0, and q must be a factor of a_n.

Example 5: List the potential rational zeros of $f(x) = 3x^3 + 14x^2 + 7x - 4$.

$f(x)$ has integer coefficients, so the Rational Zeros Theorem may be used.

List all integers p that are factors of the constant term $a_0 = -4$ and all integers q that are factors of the leading coefficient $a_3 = 3$.

$a_0 = -4$, so p: $\pm1, \pm2, \pm4$ Factors of -4

$a_3 = 3$, so q: $\pm1, \pm3$ Factors of 3

Now form all possible ratios $\frac{p}{q}$: $\pm1, \pm2, \pm4, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}$

If $f(x)$ has a rational zero, it will be from this list of 12 possibilities.

Section 4.5 – The Real Zeros of a Polynomial Function – Day 1 (continued)

Remember what the Rational Zeros Theorem says: for a polynomial function with integer coefficients, **if** there is a rational zero, it is one of those listed. There may not be any rational zeros.

All material has been taken from ***Precalculus,*** by M. Sullivan, 10th Edition