Section 4.5 – The Real Zeros of a Polynomial Function – Day 2

Find the Real Zeros of a Polynomial Function

Example 6: Find the zeros of $f(x)=3x^3+14x^2+7x-4$. Write f in factored form.

$f(x)$ is of degree 3. $\Rightarrow$ There are at most 3 real zeros.

$f(x)=3x^3+14x^2+7x-4$

$+\quad +\quad +\ \vee\ -$ $\Rightarrow$ 1 variation in sign

$f(x)$ has 1 variation in sign, so by Descartes' Rule of Signs, there is 1 positive real zero.

$f(-x)=3(-x)^3+14(-x)^2+7(-x)-4$

$=-3x^3+14x^2-7x-4$

$-\ \vee\ +\ \vee\ -\quad -$ $\Rightarrow$ 2 variations in sign

$f(-x)$ has 2 variations in sign, so by Descartes' Rule of Signs, there are 2 or 0 negative real zeros.

Determine the list of potential rational zeros:

$a_0=-4$, so p: $\pm1,\ \pm2,\ \pm4$ Factors of -4

$a_3=3$, so q: $\pm1,\ \pm3$ Factors of 3

Now form all possible ratios $\frac{p}{q}$: $\pm1,\ \pm2,\ \pm4,\ \pm\frac{1}{3},\ \pm\frac{2}{3},\ \pm\frac{4}{3}$

If $f(x)$ has a rational zero, it will be from this list of 12 possibilities.

Use synthetic division to test potential zeros:

Test 1:
$$\begin{array}{r|rrrr} 1 & 3 & 14 & 7 & -4 \\ & & 3 & 17 & 24 \\ \hline & 3 & 17 & 24 & 20 \end{array}$$
So, $f(1)=20$.

$f(1)\neq 0$. $\Rightarrow$ 1 is not a zero

Test 2:
$$\begin{array}{r|rrrr} 2 & 3 & 14 & 7 & -4 \\ & & 6 & 40 & 94 \\ \hline & 3 & 20 & 47 & 90 \end{array}$$
So, $f(2)=90$.

$f(2)\neq 0$. $\Rightarrow$ 2 is not a zero

Test $\frac{1}{3}$:
$$\begin{array}{r|rrrr} \frac{1}{3} & 3 & 14 & 7 & -4 \\ & & 1 & 5 & 4 \\ \hline & 3 & 15 & 12 & 0 \end{array}$$
$\Rightarrow f(1/3)=0$ is the remainder

$\Rightarrow \frac{1}{3}$ is a zero and

$x-\frac{1}{3}$ is a factor of $f(x)$

So, $f(x)=\left(x-\frac{1}{3}\right)(3x^2+15x+12)$

$=3\left(x-\frac{1}{3}\right)(x^2+5x+4)$

$=3\left(x-\frac{1}{3}\right)(x+1)(x+4)$

You already know $\frac{1}{3}$ is a zero of $f(x)$, find the remaining zeros:

Set $3(x+1)(x+4)=0$

$\Rightarrow\quad x+1=0$ or $x+4=0$

$x=-1$ or $x=-4$

$\Rightarrow$ The zeros of $f(x)$ are -4, -1, and $\frac{1}{3}$, all of which were on our list of potential zeros of $f(x)$.

In example 6, $f(x)=3x^3+14x^2+7x-4$

$=(x+1)(3x^2+11x-4)$, so any solution of the equation $3x^2+11x-4$ will be a solution of $f(x)$.

We call the equation $3x^2+11x-4$ a depressed equation of f.

Section 4.5 – The Real Zeros of a Polynomial Function – Day 2 (continued)

Steps for Finding the Real Zeros of a Polynomial Function:

1) Use the degree of the polynomial to determine the maximum number of real zeros.

2) Use Descartes' Rule of Signs to determine the possible number of positive real zeros and negative real zeros.

3) a) If the polynomial has integer coefficients, use the Rational Zeros Theorem to identify those rational numbers that potentially could be zeros.

   b) Use substitution, synthetic division, or long division to test each potential rational zero. Each time that a zero (and thus a factor) is found, repeat Step 3 on the depressed equation.

In attempting to find the zeros, remember to use (if possible) the factoring techniques that you already know (special products, factoring by grouping, and so on.)

Example 7: Find the real zeros of $f(x) = 3x^5 - 2x^4 - 15x^3 + 10x^2 + 12x - 8$.

1) $f(x)$ is of degree 5. $\Rightarrow$ There are at most 5 real zeros.

2) Descartes' Rule of Signs:

$f(x) = 3x^5 - 2x^4 - 15x^3 + 10x^2 + 12x - 8$

+ \/ –   – \/ +   + \/ –

$f(x)$ has 3 variations in sign $\Rightarrow$ 3 or 1 positive real zeros

$f(-x) = 3(-x)^5 - 2(-x)^4 - 15(-x)^3 + 10(-x)^2 + 12(-x) - 8$

$= -3x^5 - 2x^4 + 15x^3 + 10x^2 - 12x - 8$

–   – \/ +   + \/ –   –

$f(-x)$ has 2 variations in sign $\Rightarrow$ 2 or 0 negative real zeros

3) Potential rational zeros:

factors p of $a_0 = -8$ and factors q of $a_5 = 3$

$a_0 = -8$, so p: $\pm 1, \pm 2, \pm 4, \pm 8$

$a_5 = 3$, so q: $\pm 1, \pm 3$

Potential rational zeros $\frac{p}{q}$: $\pm 1, \pm 2, \pm 4, \pm 8, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm\frac{8}{3}$

Test 1:

1)	3	−2	−15	10	12	−8	
		3	1	−14	−4	8	$\Rightarrow f(1) = 0$ is the remainder
	3	1	−14	−4	8	0	Thus, 1 is a zero and $x - 1$ is a factor.

So, $f(x) = (x-1)(3x^4 + x^3 - 14x^2 - 4x + 8)$

$= (x-1)g(x)$

Now work with the depressed equation $g(x) = 3x^4 + x^3 - 14x^2 - 4x + 8$.

Find the zeros of $g(x) = 3x^4 + x^3 - 14x^2 - 4x + 8$.

Repeat Step 3: The potential rational zeros of $g(x)$ are still $\frac{p}{q}$: $\pm 1, \pm 2, \pm 4, \pm 8, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm\frac{8}{3}$.

Test 1:

1)	3	1	−14	−4	8	
		3	4	−10	−14	$\Rightarrow g(1) = -6$ is the remainder
	3	4	−10	−14	−6	So $g(1) \neq 0$, and $x - 1$ is not a factor.

Section 4.5 – The Real Zeros of a Polynomial Function – Day 2 (continued)

Test 2: $\begin{array}{r|rrrrr} -1 & 3 & 1 & -14 & -4 & 8 \\ & & -3 & 2 & 12 & -8 \\ \hline & 3 & -2 & -12 & 8 & 0 \end{array}$ $\Rightarrow g(-1)=0$ is the remainder. Thus, -1 is a zero and $x+1$ is a factor.

So, $f(x)=(x-1)g(x)$

$=(x-1)(x+1)(3x^3-2x^2-12x+8)$

$=(x-1)(x+1)h(x)$

The remaining zeros satisfy the new depressed equation $h(x)=3x^3-2x^2-12x+8$.

Setting $h(x)=0$:

$3x^3-2x^2-12x+8=0$, you may notice that it can be factored by grouping.

$3x^3-2x^2-12x+8=0$

$(3x^3-2x^2)+(-12x+8)=0$

$x^2(3x-2)-4(3x-2)=0$

$(x^2-4)(3x-2)=0$

$(x+2)(x-2)(3x-2)=0$

Find the remaining zeros of $f(x)$: $(x+2)(x-2)(3x-2)=0$

$\Rightarrow x+2=0$ or $x-2=0$ or $3x-2=0$

$3x=2$

So, $x=-2$, $x=2$, or $x=\frac{2}{3}$.

So, $f(x)=(x-1)(x+1)h(x)$

$=(x-1)(x+1)(3x^3-2x^2-12x+8)$

$=(x-1)(x+1)(x+2)(x-2)(3x-2)$

The real zeros of $f(x)$ are 1, -1, -2, 2, and $\frac{2}{3}$, all of which were on our list of potential zeros of $f(x)$.

The factored form of $f(x)$ is $f(x)=3(x-1)(x+1)(x+2)(x-2)\left(x-\frac{2}{3}\right)$.

You use the same procedure you used to find the zeros of a polynomial to solve a polynomial equation.

Example 8: Solve the equation $x^5-5x^4+12x^3-24x^2+32x-16=0$.

Let $f(x)=x^5-5x^4+12x^3-24x^2+32x-16$.

1) $f(x)$ is of degree 5. $\Rightarrow$ There are at most 5 real solutions.

2) Descartes' Rule of Signs:

$f(x)=x^5-5x^4+12x^3-24x^2+32x-16$

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$f(x)$ has 5 variations in sign $\Rightarrow$ 5, 3, or 1 positive real solutions

$f(-x)=(-x)^5-5(-x)^4+12(-x)^3-24(-x)^2+32(-x)-16$

$=-x^5-5x^4-12x^3-24x^2-32x-16$

$- \quad - \quad - \quad - \quad - \quad -$

$f(-x)$ has no variations in sign $\Rightarrow$ No negative real solutions

Section 4.5 – The Real Zeros of a Polynomial Function – Day 2 (continued)

3) Potential rational solutions:

factors p of $a_0 = -16$ and factors q of $a_5 = 1$

$a_0 = -16$, so p: $\pm 1,\ \pm 2,\ \pm 4,\ \pm 8,\ \pm 16$

$a_5 = 1$, so q: ± 1

Potential rational solutions $\frac{p}{q}$: $\pm 1,\ \pm 2,\ \pm 4,\ \pm 8,\ \pm 16$

You know there are no negative real solutions, so only test positive potential solutions.

Test 1:

$$\begin{array}{r|rrrrrr} 1 & 1 & -5 & 12 & -24 & 32 & -16 \\ & & 1 & -4 & 8 & -16 & 16 \\ \hline & 1 & -4 & 8 & -16 & 16 & 0 \end{array}$$

$\Rightarrow f(1) = 0$ is the remainder

Thus, 1 is a zero and $x-1$ is a factor.

So, $f(x) = (x-1)(x^4 - 4x^3 + 8x^2 - 16x + 16)$

$= (x-1)g(x)$

Now work with the depressed equation $g(x) = x^4 - 4x^3 + 8x^2 - 16x + 16 = 0$.

Find the real solutions of $g(x) = x^4 - 4x^3 + 8x^2 - 16x + 16 = 0$.

Repeat Step 3: The potential rational zeros of $g(x)$ are still $\frac{p}{q}$: $\pm 1,\ \pm 2,\ \pm 4,\ \pm 8,\ \pm 16$

Again, since you know there are no negative real solutions, test only positive potential solutions.

Test 1:

$$\begin{array}{r|rrrrr} 1 & 1 & -4 & 8 & -16 & 16 \\ & & 1 & -3 & 5 & -11 \\ \hline & 1 & -3 & 5 & -11 & 5 \end{array}$$

$\Rightarrow g(1) = 5$ is the remainder

$g(1) \neq 0 \Rightarrow 1$ is not a zero

Test 2:

$$\begin{array}{r|rrrrr} 2 & 1 & -4 & 8 & -16 & 16 \\ & & 2 & -4 & 8 & -16 \\ \hline & 1 & -2 & 4 & -8 & 0 \end{array}$$

$\Rightarrow g(2) = 0$ is the remainder

Thus, 2 is a zero and $x-2$ is a factor.

So, $f(x) = (x-1)(x^4 - 4x^3 + 8x^2 - 16x + 16)$

$= (x-1)g(x)$

$= (x-1)(x-2)(x^3 - 2x^2 + 4x - 8)$

$= (x-1)(x-2)h(x)$

Now work with the depressed equation $h(x) = x^3 - 2x^2 + 4x - 8 = 0$.

Find the real solutions of $h(x) = x^3 - 2x^2 + 4x - 8 = 0$.

Repeat Step 3: The potential rational zeros of $h(x)$ are still $\frac{p}{q}$: $\pm 1,\ \pm 2,\ \pm 4,\ \pm 8,\ \pm 16$

Again, since you know there are no negative real solutions, test only positive potential solutions.

1 was not a factor of $g(x)$, so it cannot be a factor of $h(x)$, a factor of $g(x)$.

Test 1:

$$\begin{array}{r|rrrr} 2 & 1 & -2 & 4 & -8 \\ & & 2 & 0 & 8 \\ \hline & 1 & 0 & 4 & 0 \end{array}$$

$\Rightarrow h(2) = 0$ is the remainder

Thus, 2 is a zero and $x-2$ is a factor.

Section 4.5 – The Real Zeros of a Polynomial Function – Day 2 (continued)

So, $f(x) = (x-1)(x-2)(x^3 - 2x^2 + 4x - 8)$
$= (x-1)(x-2)h(x)$
$= (x-1)(x-2)(x-2)(x^2+4)$
$= (x-1)(x-2)^2(x^2+4)$

$x^2 + 4 = 0$
$x^2 = -4$ **OR** $b^2 - 4ac = 0^2 - 4(1)(4)$
$x = \pm\sqrt{-4}$ $\quad = 0 - 16$
$\Rightarrow$ no real solution $\quad = -16$
$\quad < 0 \Rightarrow$ no real solution

Since $x^2 + 4$ has no real solution, the real solutions of $f(x) = x^5 - 5x^4 + 12x^3 - 24x^2 + 32x - 16 = 0$ are $x = 1$ and $x = 2$, where $x = 2$ is a double root, or a root of multiplicity 2.

The factor $x^2 + 4$, in $f(x)$ above, is called irreducible, because $x^2 + 4$ cannot be factored over the real numbers. In general, a quadratic factor $ax^2 + bx + c$ is **irreducible** if it cannot be factored over the real numbers – that is, if it is prime over the real numbers.

Theorem – Every polynomial function (with real coefficients) can be uniquely factored into a product of linear factors and/or irreducible quadratic factors.

You will look at this theorem again in Section 4.6 and draw several additional conclusions about the zeros of a polynomial function. Now it is worth noting that if a polynomial with real coefficients is of odd degree, it must contain at least one linear factor. This means it must have at least one real zero.

Theorem – A polynomial function (with real coefficients) of odd degree has a least one real zero.

Use the Intermediate Value Theorem

Since the graph of a polynomial function is continuous (no holes or gaps), we have the following theorem.

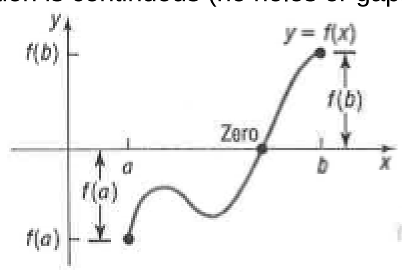

Figure 45 If $f(a) < 0$ and $f(b) > 0$, and if f is continuous, there is a zero between a and b.

Intermediate Value Theorem –
Let f denote a polynomial function. If $a < b$ and if $f(a)$ and $f(b)$ are of opposite sign, then there is at least one zero of f between a and b.

Example 9: Show that $f(x) = x^5 - 2x^3 - 3$ has a zero between 1 and 2.

Evaluate $f(x)$ at 1 and 2:

$f(1) = 1^5 - 2(1)^3 - 3$
$= 1 - 2(1) - 3$
$= 1 - 2 - 3$
$= -1 - 3$
$= -4$

$f(2) = 2^5 - 2(2)^3 - 3$
$= 32 - 2(8) - 3$
$= 32 - 16 - 3$
$= 16 - 3$
$= 13$

Because $1 < 2$, and $f(1) < 0$ and $f(2) > 0$, it follows from the Intermediate Value Theorem that the polynomial function f has at least one zero between 1 and 2.

All material has been taken from *Precalculus,* by M. Sullivan, 10th Edition