Section 4.6 – Complex Zeros; Fundamental Theorem of Algebra

In section 4.5 you found the real zeros of a polynomial function. In this section, you will find the complex zeros of a polynomial function. Finding the complex zeros of a function requires finding all zeros of the form $a+bi$. These zeros are real if $b=0$. Recall that the set of real numbers is a subset of the set of complex numbers.

A variable in the complex number system is referred to as a **complex variable**.

A **complex polynomial function f** of degree n is a complex function of the form $f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$, where $a_n, a_{n-1},\ldots,a_1,a_0$ are complex numbers, $a_n\neq 0$, n is a nonnegative integer, and x is a complex variable. a_n is called the **leading coefficient** of f. A complex number r is called a **complex zero** of the complex function f if $f(r)=0$.

Some quadratic equations have no real solutions. However, every quadratic equation in the complex number system has either a real or complex solution. The next theorem gives an extension to complex polynomials.

Fundamental Theorem of Algebra –

Every complex polynomial function $f(x)$ of degree $n\geq 1$ has at least one complex zero.

Theorem –

Every complex polynomial function $f(x)$ of degree $n\geq 1$ can be factored into n linear factors (not necessarily distinct) of the form $f(x)=a_n(x-r_1)(x-r_2)\ldots(x-r_n)$, where a_n, r_1, $r_2,\ldots$, r_n are complex numbers.
That is, every complex polynomial function of degree $n\geq 1$ has exactly n complex (not necessarily distinct) zeros.

Conjugate Pairs Theorem –

Let $f(x)$ be a complex polynomial whose coefficients are real numbers. If $r=a+bi$ is a zero of f, then the complex conjugate $\overline{r}=a-bi$ is also a zero of f.

The Conjugate Pairs Theorem tells you that for complex polynomials whose coefficients are real numbers, complex zeros occur in conjugate pairs.

Corollary –

A complex polynomial f of odd degree with real coefficients has at least one real zero.

Example 1: A polynomial f of degree 7 whose coefficients are real numbers has the zeros 6, $2i$, $3i$, and $4+2i$. Find the remaining three zeros.

Complex zeros occur as conjugate pairs, so the remaining three zeros are
$\overline{2i}=-2i$, $\overline{3i}=-3i$, and $\overline{4+2i}=4-2i$.

Example 2: Find a polynomial f of degree 5 whose coefficients are real numbers and that has the zeros 2, $-3+i$, and $4i$.

Complex zeros occur as conjugate pairs, so $\overline{-3+i}=-3-i$ and $\overline{4i}=-4i$ are also zeros.
Because of the Factor Theorem (Sec 3.5), you have
$f(x)=a(x-2)\left[x-(-3+i)\right]\left[x-(-3-i)\right](x-4i)\left[x-(-4i)\right]$. You can assume $a=1$.

$$\begin{aligned}
f(x)&=(x-2)\left[x-(-3+i)\right]\left[x-(-3-i)\right](x-4i)\left[x-(-4i)\right]\\
&=(x-2)\left[x^2-(-3-i)x-(-3+i)x+(-3+i)(-3-i)\right]\left[x^2-(-4ix)-4ix-4i(4i)\right]\\
&=(x-2)\left[x^2+3x+ix+3x-ix+9+3i-3i-i^2\right]\left[x^2+4ix-4ix-16i^2\right]\\
&=(x-2)\left[x^2+6x+9-i^2\right]\left[x^2-16i^2\right]\\
&=(x-2)\left[x^2+6x+9-(-1)\right]\left[x^2-16(-1)\right]\\
&=(x-2)\left[x^2+6x+10\right]\left[x^2+16\right]\\
&=(x-2)\left[x^4+6x^3+10x^2+16x^2+96x+160\right]\\
&=(x-2)\left[x^4+6x^3+26x^2+96x+160\right]\\
&=x^5+6x^4+26x^3+96x^2+160x-2x^4-12x^3-52x^2-192x-320\\
&=x^5+4x^4+14x^3+44x^2-32x-320
\end{aligned}$$

Section 4.6 – Complex Zeros; Fundamental Theorem of Algebra (continued)

The value of a_n in the factor theorem above does not affect the zeros of $f(x)$. It affects the graph of $f(x)$, as it causes a stretch or compression, and a reflection also occurs if $a_n < 0$.

Theorem –
Every polynomial function with real coefficients can be uniquely factored over the real numbers into a product of linear factors and/or irreducible quadratic factors.

Example 3: Finding the complex zeros of a polynomial function.
If $1+i$ is a zero of $f(x)=x^4 - x^3 - 6x^2 + 14x - 12$, find the remaining zeros.

$1+i$ is a zero $\Rightarrow$ Its complex conjugate, $1-i$, is also a zero.
So, $[x-(1+i)]$ and $[x-(1-i)]$ are factors of f.
$\Rightarrow$ $[x-(1+i)]\,[x-(1-i)]$ is a factor of f.

$$\begin{aligned}[x-(1+i)]\,[x-(1-i)] &= x^2 - (1-i)x - (1+i)x + (1+i)(1-i)\\ &= x^2 - x + ix - x - ix + 1 - i + i - i^2\\ &= x^2 - 2x + 1 - i^2\\ &= x^2 - 2x + 1 - (-1)\\ &= x^2 - 2x + 2\end{aligned}$$

Determine the remaining factors:

$$\begin{array}{r}
x^2 + x - 6 \\
x^2 - 2x + 2 \overline{\big)\, x^4 - x^3 - 6x^2 + 14x - 12} \\
\underline{-(x^4 - 2x^3 + 2x^2)} \\
x^3 - 8x^2 + 14x \\
\underline{-(x^3 - 2x^2 + 2x)} \\
-6x^2 + 12x - 12 \\
\underline{-(-6x^2 + 12x - 12)} \\
0
\end{array}$$

So, $f(x) = (x^2 - 2x + 2)(x^2 + x - 6)$
$= (x^2 - 2x + 2)(x+3)(x-2)$
$= [x-(1+i)]\,[x-(1-i)](x-(-3))(x-2)$

By the Factor Theorem, the zeros of f are $1+i$, $1-i$, -3, and 2.

Example 4: Find the complex zeros of $f(x) = 2x^4 - 7x^3 + 11x^2 - 28x + 12$.

1) $f(x)$ is of degree 4. $\Rightarrow$ There are 4 complex zeros.
2) Descartes' Rule of Signs:

$f(x) = 2x^4 - 7x^3 + 11x^2 - 28x + 12$

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$f(x)$ has 4 variations in sign $\Rightarrow$ 4, 2, or 0 positive real zeros

$f(-x) = 2(-x)^4 - 7(-x)^3 + 11(-x)^2 - 28(-x) + 12$
$= 2x^4 + 7x^3 + 11x^2 + 28x + 12$

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$f(-x)$ has no variations in sign $\Rightarrow$ $f(x)$ has no negative real zeros

Section 4.6 – Complex Zeros; Fundamental Theorem of Algebra (continued)

3) Potential rational zeros:
factors p of $a_0 = 12$ and factors q of $a_4 = 2$

$a_0 = 12$, so p: $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

$a_4 = 2$, so q: $\pm 1, \pm 2$

Potential rational zeros $\frac{p}{q}$: $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm\frac{1}{2}, \pm\frac{3}{2}$

Only consider the positive zeros, since in step 2 you've shown there are no negative real zeros.

Test 1: 1) 2 −7 11 −28 12

1	2	−7	11	−28	12
		2	−5	6	−22
	2	−5	6	−22	−10

$\Rightarrow f(1) = -10$ is the remainder

$f(1) \neq 0 \Rightarrow$ 1 is not a zero.

Test 2:

2	2	−7	11	−28	12
		4	−6	10	−36
	2	−3	5	−18	−24

$\Rightarrow f(2) = -24$ is the remainder

$f(2) \neq 0 \Rightarrow$ 2 is not a zero.

Test 3:

3	2	−7	11	−28	12
		6	−3	24	−12
	2	−1	8	−4	0

$\Rightarrow f(3) = 0$ is the remainder

Thus, 3 is a zero and $x-3$ is a factor.

So, $f(x) = (x-3)(2x^3 - x^2 + 8x - 4)$
$= (x-3)g(x)$

Now work with the depressed equation $g(x) = 2x^3 - x^2 + 8x - 4$.

Find the zeros of $g(x) = 2x^3 - x^2 + 8x - 4$.

Repeat Step 3: The potential rational zeros of $g(x)$ are still

$$\frac{p}{q}: \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm\frac{1}{2}, \pm\frac{3}{2}$$

Only consider the positive zeros, since in Step 2 you've stated there are no negative real zeros. Also, 1 and 2 were not factors of f, so they are not factors of g, since g is a factor of f. Thus, there is no need to test the potential zeros 1 and 2.

Test 1:

4	2	−1	8	−4
		8	28	144
	2	7	36	140

$\Rightarrow g(4) = 140$ is the remainder

$g(4) \neq 0 \Rightarrow$ 4 is not a zero.

Test 2:

$\frac{1}{2}$	2	−1	8	−4
		1	0	4
	2	0	8	0

$\Rightarrow g\left(\frac{1}{2}\right) = 0$ is the remainder

Thus, $\frac{1}{2}$ is a zero and $x - \frac{1}{2}$ is a factor.

Section 4.6 – Complex Zeros; Fundamental Theorem of Algebra (continued)

$$\begin{aligned}
\text{So, } f(x) &= 2x^4 - 7x^3 + 11x^2 - 28x + 12 \\
&= (x-3)(2x^3 - x^2 + 8x - 4) \\
&= (x-3)g(x) \\
&= (x-3)\left(x - \frac{1}{2}\right)(2x^2 + 8) \\
&= 2(x-3)\left(x - \frac{1}{2}\right)(x^2 + 4) \\
&= 2(x-3)\left(x - \frac{1}{2}\right)(x + 2i)(x - 2i) \\
&= 2(x-3)\left(x - \frac{1}{2}\right)\left[x - (-2i)\right](x - 2i)
\end{aligned}$$

By the Factor Theorem, the complex zeros of $f(x)$ are 3, $\frac{1}{2}$, $-2i$, and $2i$.