Section 5.1 – Composite Functions

From Section 2.1, you know functions, like numbers, can be added, subtracted, multiplied, and divided.

If f and g are functions:

Their sum, $f+g$, is the function defined by $(f+g)(x) = f(x) + g(x)$.
The domain of $(f+g)(x)$ consists of the numbers x that are in the domain of f and in the domain of g.

Their difference, $f-g$, is the function defined by $(f-g)(x) = f(x) - g(x)$.
The domain of $(f-g)(x)$ consists of the numbers x that are in the domain of f and in the domain of g.

Their product, $f \bullet g$, is the function defined by $(f \bullet g)(x) = f(x) \bullet g(x)$.
The domain of $(f \bullet g)(x)$ consists of the numbers x that are in the domain of f and in the domain of g.

Their quotient, $\frac{f}{g}$, is the function defined by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, where $g(x) \neq 0$.
The domain of $\left(\frac{f}{g}\right)(x)$ consists of the numbers x for which $g(x) \neq 0$ that are in the domain of f and in the domain of g.

Example 1: Let f and g be two functions defined as $f(x) = 5x^2$ and $g(x) = 3x - 1$. Find all four operations of f and g and the domain for each operation.

a) $(f+g)(x) = f(x) + g(x)$
$= 5x^2 + 3x - 1$
Domain: All real numbers

b) $(f-g)(x) = f(x) - g(x)$
$= 5x^2 - (3x-1)$
$= 5x^2 - 3x + 1$
Domain: All real numbers

c) $(f \bullet g)(x) = f(x) \bullet g(x)$
$= 5x^2(3x-1)$
$= 15x^3 - 5x^2$
Domain: All real numbers

d) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$
$= \frac{5x^2}{3x-1}$
Domain: All real numbers except $x = \frac{1}{3}$
OR $\left\{x \,\middle|\, x \neq \frac{1}{3}\right\}$

Form a Composite Function

Given two functions f and g, the composite function, denoted by $f \circ g$ (read as "f composed with g"), is defined by

$$(f \circ g)(x) = f(x) \circ g(x) = f(g(x)).$$

The domain of $(f \circ g)(x)$ is the set of all numbers x in the domain of g such that $g(x)$ is in the domain of f.

Look at Figure 2. Only those values of x in the domain of g for which g(x) is in the domain of f can be in the domain of $f \circ g$. The reason is that if g(x) is not in the domain of f, then f(g(x)) is not defined. Because of this, the domain of $f \circ g$ is a subset of the domain of g; the range of $f \circ g$ is a subset of the range of f.

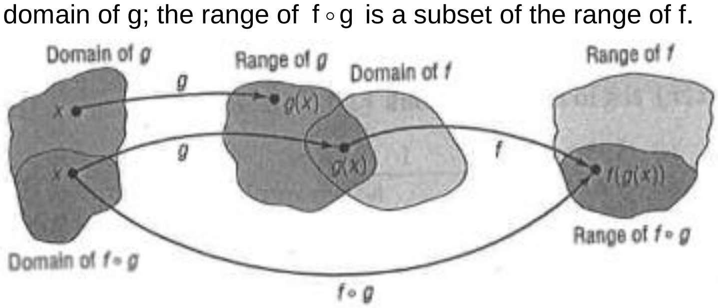

Figure 2

Section 5.1 – Composite Functions (continued)

Figure 3 provides a second illustration of the definition. Here x is the input to the function g, yielding g(x). Then g(x) is the input to the function f, yielding $f(g(x))$. Note that the "inside" function g in $f(g(x))$ is "processed" first.

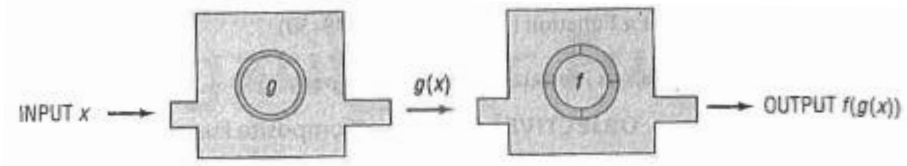

Figure 3

Example 2: Let f and g be two functions defined as $f(x) = x^2 - 1$ and $g(x) = 3x$. Find $(f \circ g)(1)$ and $(g \circ f)(2)$.

a) $(f \circ g)(x) = f(x) \circ g(x)$
$= f(g(x))$
$= (3x)^2 - 1$
$= 3^2 x^2 - 1$
$= 9x^2 - 1$

$f(g(1)) = 9(1)^2 - 1$
$= 9 - 1$
$= 8$

OR

Alternate Method

$g(x) = 3x$
$\Rightarrow g(1) = 3(1)$
$= 3$

So, $f(g(1)) = f(3)$
$= 3^2 - 1$
$= 9 - 1$
$= 8$

b) $(g \circ f)(x) = g(x) \circ f(x)$
$= g(f(x))$
$= 3(x^2 - 1)$
$= 3x^2 - 3$

$g(f(2)) = 3(2)^2 - 3$
$= 3(4) - 3$
$= 12 - 3$
$= 9$

OR

Alternate Method

$f(x) = x^2 - 1$
$\Rightarrow f(2) = 2^2 - 1$
$= 4 - 1$
$= 3$

So, $g(f(2)) = g(3)$
$= 3(3)$
$= 9$

Find the Domain of a Composite Function

To find the domain of $(f \circ g)(x) = f(g(x))$, remember that

1) $g(x)$ must be defined, so exclude any value of x that is not in the domain of $g(x)$.
2) $f(g(x))$ must be defined, so any x for which $g(x)$ is not in the domain of f must be excluded.

Example 3: Find the domain of $(f \circ g)(x)$ if $f(x) = \frac{1}{x-3}$ and $g(x) = \frac{2}{x+5}$.

The domain of $g(x)$ is $\{x | x \neq -5\}$.

$\Rightarrow$ You must exclude -5 from the domain of $(f \circ g)(x)$.

The domain of $f(x)$ is $\{x | x \neq 3\}$. $(f \circ g)(x) = f(g(x))$, so $g(x) \neq 3$.

Find where $g(x) = 3$. Set $g(x) = 3$.

$$\frac{2}{x+5} = 3$$
$$2 = 3(x+5)$$
$$2 = 3x + 15$$
$$-13 = 3x$$
$$x = \frac{-13}{3}$$

Thus, the domain of $(f \circ g)(x)$ is $\left\{ x \,\middle|\, x \neq -5 \text{ and } x \neq \frac{-13}{3} \right\}$.

Section 5.1 – Composite Functions (continued)

Example 4: Given $f(x)=\dfrac{5}{x-2}$ and $g(x)=\dfrac{-3}{x+4}$.

a) Find the domain of $f\circ g$ and then the function $(f\circ g)(x)$.

b) Find the domain of $g\circ f$ and then the function $(g\circ f)(x)$.

a) The domain of $g(x)$ is $\{x|x\neq -4\}$.
The domain of $f(x)$ is $\{x|x\neq 2\}$.
$(f\circ g)(x)=f(g(x))$, so $g(x)\neq 2$.
Find where $g(x)=2$. Set $g(x)=2$.

$$\frac{-3}{x+4}=2$$
$$-3=2(x+4)$$
$$-3=2x+8$$
$$-11=2x$$
$$x=\frac{-11}{2}$$

So, the domain of $(f\circ g)(x)$ is $\left\{x \,\middle|\, x\neq -4 \text{ and } x\neq \dfrac{-11}{2}\right\}$.

$$f(g(x))=f\left(\frac{-3}{x+4}\right)=\frac{5}{\frac{-3}{x+4}-2}=\frac{5}{\frac{-3}{x+4}-\frac{2(x+4)}{x+4}}=\frac{5}{\frac{-3}{x+4}-\frac{2x+8}{x+4}}=\frac{5}{\frac{-2x-11}{x+4}}=\frac{5(x+4)}{-2x-11}=\frac{5x+20}{-2x-11}$$

b) Given $f(x)=\dfrac{5}{x-2}$ and $g(x)=\dfrac{-3}{x+4}$.
The domain of $f(x)$ is $\{x|x\neq 2\}$.
The domain of $g(x)$ is $\{x|x\neq -4\}$.
$(g\circ f)(x)=g(f(x))$, so $f(x)\neq -4$.
Find where $f(x)=-4$. Set $f(x)=-4$.

$$\frac{5}{x-2}=-4$$
$$5=-4(x-2)$$
$$5=-4x+8$$
$$-3=-4x$$
$$x=\frac{3}{4}$$

So, the domain of $(g\circ f)(x)$ is $\left\{x \,\middle|\, x\neq 2 \text{ and } x\neq \dfrac{3}{4}\right\}$.

$$g(f(x))=g\left(\frac{5}{x-2}\right)=\frac{-3}{\frac{5}{x-2}+4}=\frac{-3}{\frac{5}{x-2}+\frac{4(x-2)}{x-2}}=\frac{-3}{\frac{5}{x-2}+\frac{4x-8}{x-2}}=\frac{-3}{\frac{4x-3}{x-2}}=\frac{-3(x-2)}{4x-3}=\frac{-3x+6}{4x-3}$$

In general, $(f\circ g)(x)\neq(g\circ f)(x)$.

Section 5.1 – Composite Functions (continued)

Example 5: Given $f(x)=5x+6$ and $g(x)=\frac{1}{5}(x-6)$. Find $(f\circ g)(x)$ and $(g\circ f)(x)$.

$$
\begin{aligned}
(f\circ g)(x) &= f(g(x)) \\
&= f\left(\frac{1}{5}(x-6)\right) \\
&= 5\left(\frac{1}{5}(x-6)\right)+6 \\
&= (x-6)+6 \\
&= x
\end{aligned}
\qquad
\begin{aligned}
(g\circ f)(x) &= g(f(x)) \\
&= \frac{1}{5}\big((5x+6)-6\big) \\
&= \frac{1}{5}(5x) \\
&= x
\end{aligned}
$$

So, in this example, $(f\circ g)(x)=(g\circ f)(x)$.

In Section 5.2, you will see that there is an important relationship between functions f and g for which $(f\circ g)(x)=(g\circ f)(x)=x$.

Calculus Application

Some techniques in calculus require the ability to determine the components of a composite function.

Example 6: The function $H(x)=\sqrt{3x^2+2}$ is the composition of functions f and g. Find functions f and g such that $(f\circ g)(x)=H(x)$.

Let $f(x)=\sqrt{x}$ and $g(x)=3x^2+2$.

Then
$$
\begin{aligned}
H(x) &= (f\circ g)(x) \\
&= f(g(x)) \\
&= f\left(3x^2+2\right) \\
&= \sqrt{3x^2+2}
\end{aligned}
$$

Functions f and g in Example 5 are not unique, but usually there is an obvious, or "natural", selection for f and g that comes to mind first.

You could also let $f(x)=\sqrt{x+2}$ and $g(x)=3x^2$.

Then
$$
\begin{aligned}
H(x) &= (f\circ g)(x) \\
&= f(g(x)) \\
&= f\left(3x^2\right) \\
&= \sqrt{3x^2+2}
\end{aligned}
$$