## Section 5.2 – One-to-One Functions; Inverse Functions

Section 2.1 presented four different ways to represent a function: 1) a map, 2) a set of ordered pairs, 3) a graph, and 4) an equation. For example, Figures 6 and 7 illustrate two different functions represented as mappings. The function in Figure 6 shows correspondence between states and their populations (in millions). The function in Figure 7 shows a correspondence between animals and life expectancies (in years).

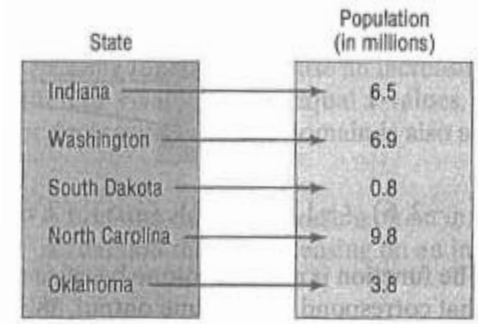


**Figure 6**

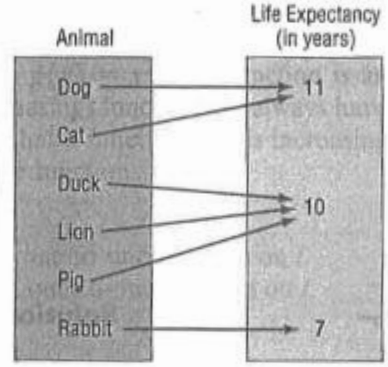


**Figure 7**

Suppose several people are asked to name a state that has a population of 0.8 million based on the function in Figure 6. Everyone will respond "South Dakota." Now, if the same people are asked to name an animal whose life expectancy is 11 years based on the function in Figure 7, some may respond "dog," while others may respond "cat." What is the difference between the functions in Figures 6 and 7? In Figure 6, no two elements in the domain correspond to the same element in the range. In Figure 7, this is not the case: Different elements in the domain correspond to the same element in the range. Functions such as the one in Figure 6 are given a special name.

Definition: A function is **one-to-one** if any two different inputs in the domain correspond to two different outputs in the range. That is, if $x_1$ and $x_2$ are two different inputs of a function, then f is one-to-one if $f(x_1) \neq f(x_2)$.

Put another way, a function f is one-to-one if no y in the range is the image of more than one x in the domain A function is not one-to-one if any two (or more) different elements in the domain correspond to the same element in the range. So, the function in Figure 7 is not one-to-one because two different elements in the domain, dog and cat, both correspond to 11 (and also because three different elements in the domain correspond to 10). So, a function is not one-to-one if two different inputs correspond to the same output.

Figure 8 illustrates the distinction among one-to-one functions, functions that are not one-to-one, and relations that are not functions.

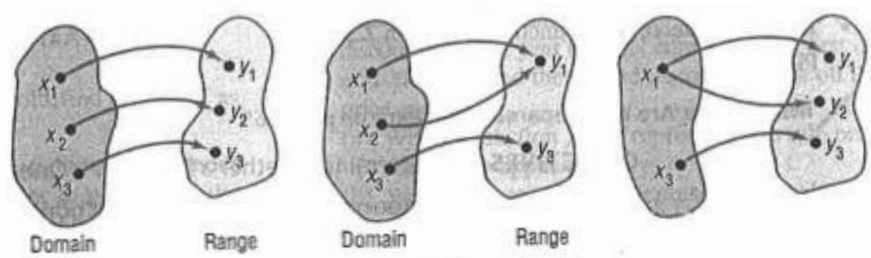
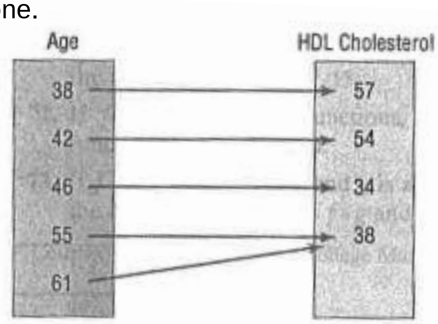



**Figure 8** (a) One-to-one function: Each $x$ in the domain has one and only one image in the range. (b) Not a one-to-one function: $y_1$ is the image of both $x_1$ and $x_2$. (c) Not a function: $x_1$ has two images, $y_1$ and $y_2$.

Example 1: Determine whether the following functions are one-to-one.

a) For the following function, the domain represents the ages of five males, and the range represents their HDL (good) cholesterol scores (mg/dL).




The function is not one-to-one because there are different inputs, 55 and 61, that correspond to the same output, 38.

b) $f = \{(-3, -24), (2, -5), (0, 11), (4, 17)\}$.

The function f is one-to-one because no two distinct inputs correspond to the same output.

## Section 5.2 – One-to-One Functions; Inverse Functions (continued)

For functions defined by an equation $y = f(x)$ and for which the graph of f is known, there is a simple test, called the **horizontal-line test**, to determine whether f is one-to-one.

Theorem: Horizontal-Line Test

If every horizontal line intersects the graph of a function f in at most one point, then f is one-to-one.

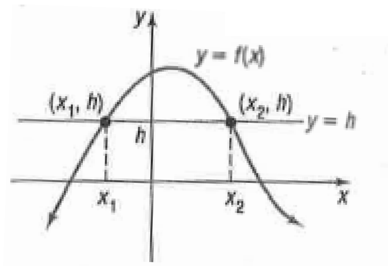
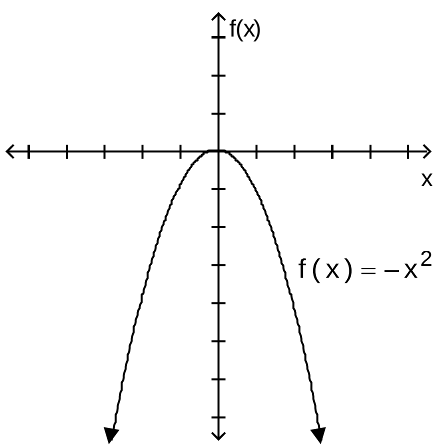



**Figure 9**
$f(x_1) = f(x_2) = h$ and $x_1 \neq x_2$;
f is not a one-to-one function.

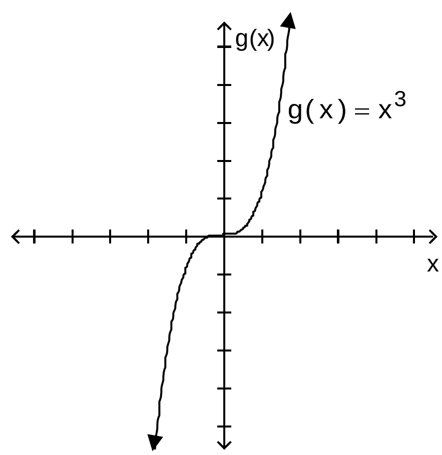
The reason why this test works can be seen in Figure 9, where the horizontal line $y = h$ intersects the graph at two distinct points, $(x_1, h)$ and $(x_2, h)$. Since h is the image of both $x_1$ and $x_2$ and $x_1 \neq x_2$, f is not one-to-one.

Based on Figure 9, you can state the horizontal line test in another way: If the graph of any horizontal line intersects the graph of a function f at more than one point, then f is not one-to-one.

Example 2: Use the graph to determine if the function is one-to-one.




One-to-one? Yes or No




One-to-one? Yes or No

Theorem –

A function that is increasing on an interval I is a one-to-one function on I.
A function that is decreasing on an interval I is a one-to-one function on I.

This theorem is true because an increasing (or decreasing) function will always have different y-values for unequal x-values, so it follows that a function that is increasing (or decreasing) on an interval is also a one-to-one function on the interval.

## Section 5.2 – One-to-One Functions; Inverse Functions (continued)

### Determine the Inverse of a Function Defined by a Map or a Set of Ordered Pairs

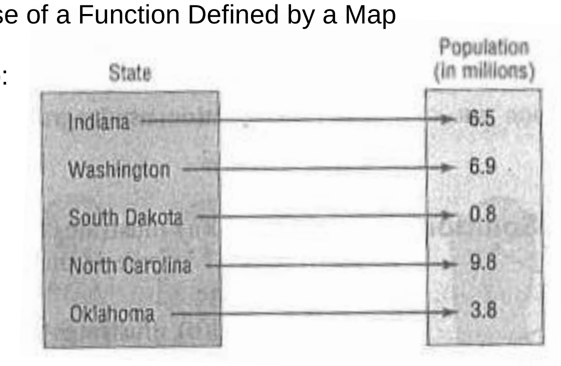
Definition: Suppose that f is a one-to-one function. Then, corresponding to each x in the domain of f, there is exactly one y in the range (because f is a function); and corresponding to each y in the range of f, there is exactly one x in the domain (because f is one-to-one). The correspondence from the range of f back to the domain of f is called the **inverse function of f**. The symbol $f^{-1}$ is used to denote the inverse function of f.

In other words, if f is a one-to-one function so that the input 4 corresponds to the output 8, then in the inverse function $f^{-1}$, the input 8 will correspond to the output 4.

We will look at how to find inverses for all four representations of functions: 1) maps, 2) sets of ordered pairs, 3) graphs, and 4) equations. We begin with finding inverses of functions represented by maps or sets of ordered pairs.

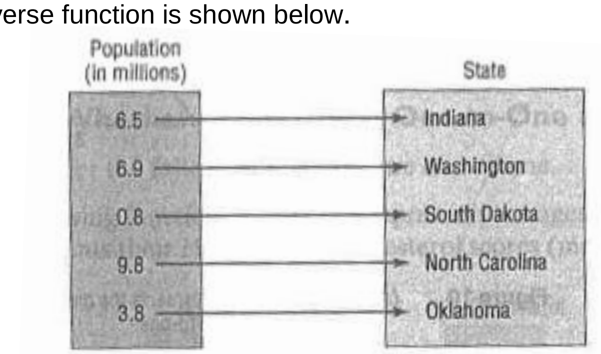
Example 3: Find the Inverse of a Function Defined by a Map

Given the map:




Let the domain of the function represent certain states, and let the range represent the states' populations (in millions). Find the domain and the range of the inverse function.

Solution: The function is one-to-one. To find the inverse function, interchange the elements in the domain with the elements in the range. For example, the function receives as input Indiana and output 6.5 million. So, the inverse receives as input 6.5 million and output Indiana. The inverse function is shown below.




The domain of the inverse function is $\{0.8, 3.8, 6.5, 6.9, 9.8\}$. The range of the inverse function is {Indiana, North Carolina, Oklahoma, South Dakota, Washington}.

If the function f is the set of ordered pairs $(x, y)$, then the inverse function of f, denoted $f^{-1}$, is the set of ordered pairs $(y, x)$.

So, to find the inverse, interchange the elements in the domain with the elements in the range.

Example 4: Find the Inverse of a Function Defined by a Set of Ordered Pairs

Find the inverse of the function $f = \{(-4, -20), (-3, -15), (0, 0), (2, 10)\}$.

The inverse of $f = \{(-20, -4), (-15, -3), (0, 0), (10, 2)\}$.

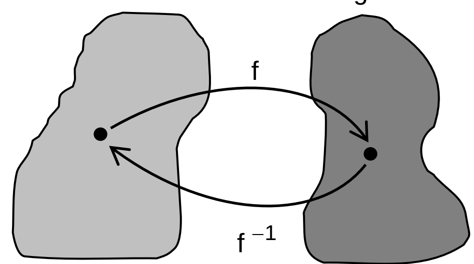
## Section 5.2 – One-to-One Functions; Inverse Functions (continued)

Remember, if f is a one-to-one function, it has an inverse function, $f^{-1}$. Be careful with the notation, as the $-1$ in $f^{-1}$ is not an exponent. That is, $f^{-1}$ does not mean the reciprocal of f ; $f^{-1}(x)$ is not equal to $\frac{1}{f(x)}$.

Let f denote a one-to-one function $y = f(x)$. The inverse of f, denoted by $f^{-1}$, is a function such that $f^{-1}(f(x)) = x$ for every x in the domain of f, and $f\left(f^{-1}(x)\right) = x$ for every x in the domain of $f^{-1}$.

Domain of f = Range of $f^{-1}$
Range of f = Domain of $f^{-1}$

i.e., What f does, $f^{-1}$ undoes, and vice versa.




Example 5: Verify that the inverse of $f(x) = 4x + 2$ is $f^{-1}(x) = \frac{1}{4}(x-2)$.

You need to show that $f\left(f^{-1}(x)\right) = x$ and $f^{-1}\left(f(x)\right) = x$.

$$f\left(f^{-1}(x)\right) = f\left(\frac{1}{4}(x-2)\right) = 4\left(\frac{1}{4}(x-2)\right) + 2 = (x-2) + 2 = x$$

$$f^{-1}\left(f(x)\right) = f^{-1}\left(4x+2\right) = \frac{1}{4}\left((4x+2)-2\right) = \frac{1}{4}(4x) = x$$

Since $f\left(f^{-1}(x)\right) = x$ and $f^{-1}\left(f(x)\right) = x$, $f(x)$ and $f^{-1}(x)$ are inverse functions.

## Obtain the Graph of the Inverse Function from the Graph of the Function

Suppose that $(a, b)$ is a point on the graph of a one-to-one function f defined by $y = f(x)$. Then $b = f(a)$. This means that $a = f^{-1}(b)$, so $(b, a)$ is a point on the graph of the inverse function $f^{-1}$. The relationship between the point $(a, b)$ on f and the point $(b, a)$ on $f^{-1}$ is shown in Figure 13. The line segment with endpoints $(a, b)$ and $(b, a)$ is perpendicular to the line $y = x$ and is bisected by the line $y = x$. It follows that the point $(b, a)$ on $f^{-1}$ is the reflection about the line $y = x$ of the point $(a, b)$ on f.

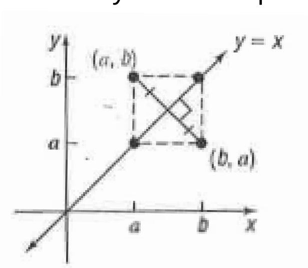


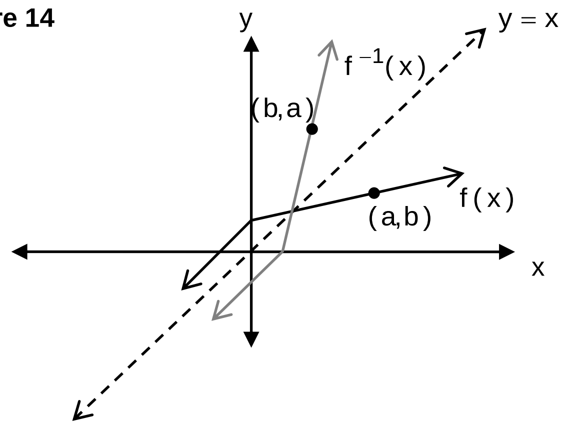

Figure 13

## Section 5.2 – One-to-One Functions; Inverse Functions (continued)

Theorem: The graph of a one-to-one function f and the graph of its inverse function $f^{-1}$ are symmetric with respect to the line $y = x$.

So, if $(a, b)$ is a point on the graph of f, then $(b, a)$ is a point on the graph of $f^{-1}$. Figure 14 illustrates this result. Once the graph of f is known, the graph of $f^{-1}$ may be obtained by reflecting the graph of f about the line $y = x$.

**Figure 14**




## Find the Inverse of a Function Defined by an Equation

The fact that the graphs of a one-to-one function f and its inverse function $f^{-1}$ are symmetric with respect to the line $y = x$ tells you more. It says that you can obtain $f^{-1}$ by interchanging the roles of x and y in f. Look again at Figure 14. If f is defined by the equation $y = f(x)$ then $f^{-1}$ is defined by the equation $x = f(y)$. The equation $x = f(y)$ defines $f^{-1}$ *implicitly*. If you can solve this equation for y, you will have the *explicit* form of $f^{-1}$, that is, $y = f^{-1}(x)$.

To find the inverse of a function $f(x)$:

1) Replace $f(x)$ with y
2) Interchange the x and y variables
3) Solve the equation for y
4) Replace y with $f^{-1}(x)$

Example 6: Find the inverse of $f(x) = 3x + 5$. Give the domain and range of f and $f^{-1}$.

Since f is a linear, increasing function, you know that f is one-to-one. So, $f^{-1}$ is a function.
Find $f^{-1}$:

$$f(x) = 3x + 5$$
$$y = 3x + 5$$
$$x = 3y + 5$$
$$x - 5 = 3y$$
$$y = \frac{x-5}{3}$$
$$f^{-1}(x) = \frac{x-5}{3}$$

Check: $f\left(f^{-1}(x)\right) = x$? AND $f^{-1}\left(f(x)\right) = x$?

$$f\left(f^{-1}(x)\right) = f\left(\frac{x-5}{3}\right) = 3\left(\frac{x-5}{3}\right) + 5 = (x-5) + 5 = x$$

$$f^{-1}\left(f(x)\right) = f^{-1}(3x+5) = \frac{(3x+5)-5}{3} = \frac{3x}{3} = x$$

Domain of f = Range of $f^{-1}$
= All real numbers

Range of f = Domain of $f^{-1}$
= All real numbers

## Section 5.2 – One-to-One Functions; Inverse Functions (continued)

### Summary

1. If a function f is one-to-one, then it has an inverse function $f^{-1}$.
2. Domain of f = Range of $f^{-1}$; Range of f = Domain of $f^{-1}$.
3. To verify that $f^{-1}$ is the inverse of f, show that $f^{-1}(f(x)) = x$ for every x in the domain of f and that $f(f^{-1}(x)) = x$ for every x in the domain of $f^{-1}$.
4. The graphs of f and $f^{-1}$ are symmetric with respect to the line $y = x$.