Section 5.3 – Exponential Functions

Recall that in the expression a^n, *a* is called the base and n is called the power or exponent. Base *a* is a positive real number and the exponent n is a rational number.

Theorem – Law of Exponents

If s, t, a, and b are real numbers with $a > 0$ and $b > 0$, then

$$a^s\left(a^t\right) = a^{s+t} \qquad (ab)^s = a^s(b^s) \qquad 1^s = 1$$

$$(a^s)^t = a^{st} \qquad a^{-s} = \frac{1}{a^s} = \left(\frac{1}{a}\right)^s \qquad a^0 = 1$$

An **exponential function** is a function of the form $f(x) = Ca^x$, where a is a positive real number ($a > 0$), $a \neq 1$, and $C \neq 0$ is a real number. The domain of f is the set of all real numbers. The base a is the **growth factor**, and, because $f(0) = Ca^0 = C$, C is called the **initial value**.

In the definition of an exponential function, the base $a = 1$ is excluded because this function is the constant function $f(x) = C(1^x) = C$. Bases that are negative are also excluded; otherwise, many values of x would have to be excluded from the domain, such as $x = \frac{1}{2}$. (If base $a = -2$ and $x = \frac{1}{2}$, then $(-2)^x = \sqrt{-2}$ and $\sqrt{-2}$ is undefined in the real number system.)

Transformations (vertical shifts, horizontal shifts, reflections, and so on) of a function of the form $f(x) = Ca^x$ also represent exponential functions. Some examples of exponential functions are $f(x) = 3^x$, $g(x) = \left(\frac{1}{5}\right)^x + 3$, and $H(x) = 4(3^{x-2})$.

For each function, note that the base of the exponential expression is a constant and the exponent contains a variable.

In the function $f(x) = 4(2^x)$, notice that the ratio of consecutive outputs is constant for 1-unit increases in the input. This ratio equals the constant 2, the base of the exponential functions. In other words,

$$\frac{f(1)}{f(0)} = \frac{4(2^1)}{4(2^0)} = \frac{4(2)}{4(1)} = \frac{4(2)}{4} = 2$$

$$\frac{f(2)}{f(1)} = \frac{4(2^2)}{4(2^1)} = \frac{4(4)}{4(2)} = \frac{4}{2} = 2$$

$$\frac{f(3)}{f(2)} = \frac{4(2^3)}{4(2^2)} = \frac{4(8)}{4(4)} = \frac{8}{4} = 2$$

and so on

This leads to the following theorem.

Theorem: For an exponential function $f(x) = Ca^x$, $a > 0$, $a \neq 1$, and $C \neq 0$, if x is any real number, then

$$\frac{f(x+1)}{f(x)} = a \quad \text{or} \quad f(x+1) = af(x)$$

Proof:
$$\frac{f(x+1)}{f(x)} = \frac{Ca^{x+1}}{Ca^x} = a^{x+1}a^{-x} = a^{x+1-x} = a^1 = a$$

Thus for 1-unit changes in the input x of an exponential function $f(x) = Ca^x$, the ratio of consecutive outputs is the constant a.

Section 5.3 – Exponential Functions (continued)

Example 1: Identifying Linear or Exponential Functions

Determine whether the given function is linear, exponential, or neither. For those that are linear, find a linear function that models the data. For those that are exponential, find an exponential function that models the data.

(a)

x	y
−1	5
0	2
1	−1
2	−4
3	−7

(b)

x	y
−1	32
0	16
1	8
2	4
3	2

(c)

x	y
−1	2
0	4
1	7
2	11
3	16

Solution: For each function, compute the average rate of change of y with respect to x and the ratio of consecutive outputs. If the average rate of change is constant, then the function is linear. If the ratio of consecutive outputs is constant, then the function is exponential.

Table 2

x	y	Average Rate of Change	Ratio of Consecutive Outputs
−1	5		
		$\frac{\Delta y}{\Delta x} = \frac{2-5}{0-(-1)} = -3$	$\frac{2}{5}$
0	2		
		$\frac{-1-2}{1-0} = -3$	$\frac{-1}{2} = -\frac{1}{2}$
1	−1		
		$\frac{-4-(-1)}{2-1} = -3$	$\frac{-4}{-1} = 4$
2	−4		
		$\frac{-7-(-4)}{3-2} = -3$	$\frac{-7}{-4} = \frac{7}{4}$
3	−7		

(a)

x	y	Average Rate of Change	Ratio of Consecutive Outputs
−1	32		
		$\frac{\Delta y}{\Delta x} = \frac{16-32}{0-(-1)} = -16$	$\frac{16}{32} = \frac{1}{2}$
0	16		
		−8	$\frac{8}{16} = \frac{1}{2}$
1	8		
		−4	$\frac{4}{8} = \frac{1}{2}$
2	4		
		−2	$\frac{2}{4} = \frac{1}{2}$
3	2		

(b)

x	y	Average Rate of Change	Ratio of Consecutive Outputs
−1	2		
		$\frac{\Delta y}{\Delta x} = \frac{4-2}{0-(-1)} = 2$	2
0	4		
		3	$\frac{7}{4}$
1	7		
		4	$\frac{11}{7}$
2	11		
		5	$\frac{16}{11}$
3	16		

(c)

Section 5.3 – Exponential Functions (continued)

a) See Table 2(a). The average rate of change for every 1-unit increase in x is -3. Therefore, the function is a linear function. In a linear function the average rate of change is the slope m, so $m=-3$. The y-intercept b is the value of the function at $x=0$, so $b=2$. The linear function that models the data is $f(x)=mx+b$

$$f(x)=-3x+2.$$

b) See Table 2(b). For this function, the average rate of change from -1 to 0 is -16, and the average rate of change from 0 to 1 is -8. Because the average rate of change is not constant, the function is not a linear function. The ratio of consecutive outputs for a 1-unit increase in the inputs is a constant, $\frac{1}{2}$. Because the ratio of consecutive outputs is constant, the function is an exponential function with a growth factor $a=\frac{1}{2}$. The initial value C of the exponential function is $C=16$, the value of the function at 0. Therefore, the exponential function that models the data is $g(x)=Ca^x$

$$g(x)=16\left(\frac{1}{2}\right)^x.$$

c) See Table 2(c). For this function, the average rate of change from -1 to 0 is 2, and the average rate of change from 0 to 1 is 3. Because the average rate of change is not constant, the function is not a linear function. The ratio of consecutive outputs from -1 to 0 is 2, and the ratio of consecutive outputs from 0 to 1 is $\frac{7}{4}$. Because the ratio of consecutive outputs is not a constant, the function is not an exponential function.

Graph Exponential Functions

We can break up the exponential functions into two types:
those with a base > 1 and those with a base between 0 and 1.

Example 2: An exponential function with a base > 1

Graph the exponential function $f(x)=3^x$.

Domain of $f(x)=$ All real numbers

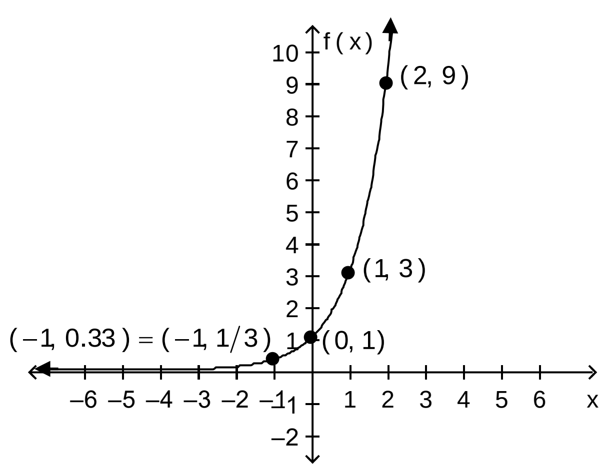
Range of $f(x)=(0,\infty)$, since $3^x>0$ for all x. $\Rightarrow$ No x-intercepts.

The y-intercept is $(0, 1)$.

As $x\to-\infty$, $f(x)$ gets closer and closer to 0 $\Rightarrow$ The x-axis, or $y=0$, is an asymptote.

As $x\to\infty$, $f(x)=3^x$ grows rapidly. For any $x_1\neq x_2$, $f(x_1)\neq f(x_2)$. So, f is an increasing function, and it is one-to-one.

x	$f(x)=3^x$
-10	$3^{-10}\approx .0000169$
-5	$3^{-5}\approx .0041$
-2	$3^{-2}\approx .1111$
-1	$3^{-1}\approx .3333$
0	$3^0=1$
1	$3^1=3$
2	$3^2=9$
5	$3^5=243$
10	$3^{10}=59049$

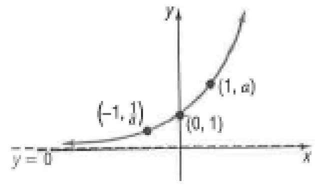


Section 5.3 – Exponential Functions (continued)

In general, exponential functions with a base larger than 1 have the following characteristics:

For $f(x) = a^x$, where $a > 1$:

- Domain: $(-\infty,\ \infty)$
- Range: $(0,\ \infty)$
- x-intercepts: none
- y-intercepts: one $(0,\ 1)$
- Horizontal asymptote: the x-axis, or $y = 0$, as $x \to -\infty$.
- No vertical asymptotes.
- $f(x)$ is an increasing function.
- $f(x)$ is a one-to-one function.
- The graph of $f(x)$ passes through the points $\left(-1,\ \frac{1}{a}\right)$, $(0,\ 1)$, and $(1,\ a)$.
- The graph of $f(x)$ rises rapidly as $x \to \infty$.
- The graph is smooth and continuous, with no corners or gaps.



Example 3: An exponential function with 0 < base < 1

Graph the exponential function $f(x) = \left(\frac{1}{3}\right)^x$.

Domain of $f(x) =$ All real numbers

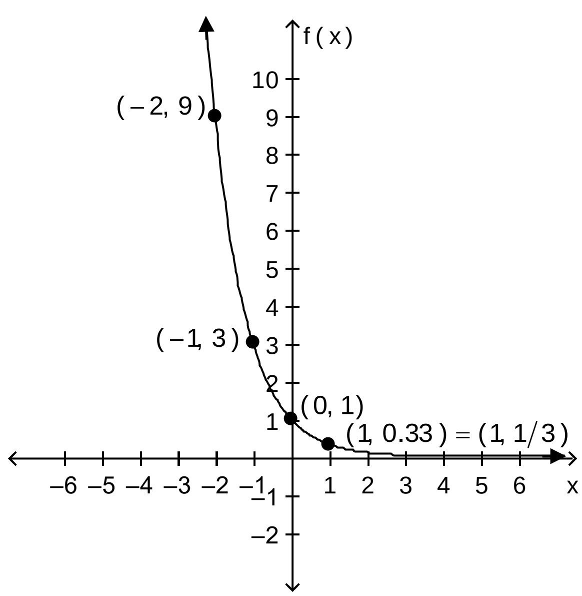
Range of $f(x) = (0, \infty)$, since $\left(\frac{1}{3}\right)^x > 0$ for all x. $\Rightarrow$ No x-intercepts.

The y-intercept is $(0, 1)$.

As $x \to \infty$, $f(x)$ gets closer and closer to 0 $\Rightarrow$ The x-axis, or $y = 0$, is an asymptote.

As $x \to -\infty$, $f(x) = \left(\frac{1}{3}\right)^x$ grows rapidly. For any $x_1 \ne x_2$, $f(x_1) \ne f(x_2)$. So, f is a decreasing function, and it is one-to-one.

x	$f(x) = \left(\frac{1}{3}\right)^x$
-10	$\left(\frac{1}{3}\right)^{-10} = 3^{10} = 59049$
-5	$\left(\frac{1}{3}\right)^{-5} = 3^5 = 243$
-2	$\left(\frac{1}{3}\right)^{-2} = 3^2 = 9$
-1	$\left(\frac{1}{3}\right)^{-1} = 3^1 = 3$
0	$\left(\frac{1}{3}\right)^0 = 1$
1	$3^{-1} \approx .3333$
2	$3^{-2} \approx .1111$
5	$3^{-5} \approx .0041$
10	$3^{-10} \approx .0000169$

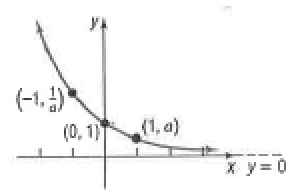


Section 5.3 – Exponential Functions (continued)

In general, exponential functions with a base between 0 and 1 have the following characteristics:

For $f(x) = a^x$, where $0 < a < 1$:

- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- x-intercepts: none
- y-intercepts: one $(0, 1)$
- Horizontal asymptote: the x-axis, or $y = 0$, as $x \to \infty$.
- No vertical asymptotes.
- $f(x)$ is a decreasing function.
- $f(x)$ is a one-to-one function.
- The graph of $f(x)$ passes through the points $\left(-1, \frac{1}{a}\right)$, $(0, 1)$, and $(1, a)$.
- The graph of $f(x)$ rises rapidly as $x \to -\infty$.
- The graph is smooth and continuous, with no corners or gaps.



Transformations (shifting, compressing, stretching, and reflecting) may be applied to basic exponential functions.

Example 4: Describe the graph of $f(x) = \left(\frac{1}{3}\right)^x + 4$.

The graph is similar to the graph in Example 3, but this function transforms the graph in Example 3 by shifting the graph up 4 units.

So, Range: $(4, \infty)$

HA: $y = 4$

The graph of $f(x)$ passes through the points $(-1, 7)$, $(0, 5)$, and $\left(1, 4\frac{1}{3}\right)$.

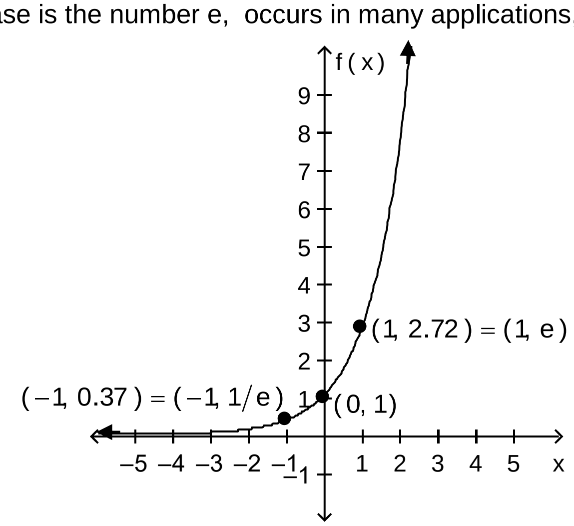
Define the number e

Many problems that occur in nature require the use of an exponential function whose base is a certain irrational number, symbolized by the letter *e*.

The **number e** is defined as the number that the expression $\left(1+\frac{1}{n}\right)^n$ approaches as $n \to \infty$. In calculus, this is expressed using limit notation as $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. As n becomes very large $(n \to \infty)$, the value of $\left(1+\frac{1}{n}\right)^n$ approaches the number $2.71828\ldots$, named e. e is an irrational number like π, so its decimal expansion continues forever without repeating patterns.

The exponential function $f(x) = e^x$, whose base is the number e, occurs in many applications, so it is known as the exponential function.

The graph of $f(x) = e^x$ is as follows:



Section 5.3 – Exponential Functions (continued)

Solve Exponential Equations

Equations that involve terms of the form a^x , where $a > 0$ and $a \neq 1$, are referred to as **exponential equations**. Such equations can sometimes be solved by appropriately applying the Laws of Exponents and the following property:

If $a^u = a^v$, then $u = v$.

This property is a consequence of the fact that exponential functions are one-to-one. To use this property, each side of the equality must be written with the same base.

In other words, this property says that when two exponential expressions with same base are equal, then their exponents are equal.

Example 5: Solving Exponential Equations

Solve each exponential equation: a) $4^{x+2} = 1024$ b) $3^{x-2} = 9^{x+5}$

a) $4^{x+2} = 1024$

$4^{x+2} = 4^5$ same base

$\Rightarrow x + 2 = 5$

$x = 3$

The solution set is $\{3\}$.

b) $3^{x-2} = 9^{x+5}$

$3^{x-2} = (3^2)^{(x+5)}$

$3^{x-2} = 3^{2x+10}$ same base

$\Rightarrow x - 2 = 2x + 10$

$-2 = x + 10$

$x = -12$

The solution set is $\{-12\}$.

Example 6: Solving an Exponential Equation

Solve: $e^{-x^2} = e^{4x-12}$

$e^{-x^2} = e^{4x-12}$ same base

$\Rightarrow -x^2 = 4x - 12$

$0 = x^2 + 4x - 12$

$0 = (x + 6)(x - 2)$

$\Rightarrow x + 6 = 0$ or $x - 2 = 0$

$x = -6$ or $x = 2$

The solution set is $\{-6, 2\}$.