Section 5.4 – Logarithmic Functions – Day 1

The logarithmic function with base a, where $a > 0$ and $a \neq 1$, is denoted by $y = \log_a x$ (read as "y is the logarithm with base a of x") and is defined by $y = \log_a x$ if and only if $x = a^y$. The domain of the function $y = \log_a x$ is $\{x \mid x > 0\}$.

As this definition illustrates, **a logarithm is a name for a certain exponent**. So $\log_a x$ represents the exponent to which *a* must be raised to obtain *x*. So, when you need to evaluate $\log_a x$, think to yourself "a raised to what power gives me x?"

Change Exponential Statements to Logarithmic Statements and Logarithmic Statements to Exponential Statements

You should be able to change from exponential to logarithmic form and from logarithmic to exponential form.

Example 1: Change each exponential statement to an equivalent logarithmic statement.

a) If $a^5 = 32$, then $\log_a 32 = 5$.

b) If $(6.4)^2 = x$, then $\log_{6.4} x = 2$.

Change each logarithmic statement to an equivalent exponential statement.

a) If $\log_8 5 = x$, then $8^x = 5$.

b) If $\log_x 7 = 4$, then $x^4 = 7$.

Evaluate Logarithmic Expressions

To find the exact value of a logarithm, write the logarithm in exponential notation using the fact that $y = \log_a x$ is equivalent to $a^y = x$, and use the fact that if $a^x = a^y$, then $x = y$.

Example 2: Find the exact value of $\log_3 81$.

To evaluate $\log_3 81$, think "3 raised to what power yields 81?"

Then, let $y = \log_3 81$.

The exponential form is $3^y = 81$.

$3^y = 3^4$

$\Rightarrow y = 4$

Thus, $\log_3 81 = 4$.

Determine the Domain of a Logarithmic Function

The logarithmic function $y = \log_a x$ has been defined as the inverse of the exponential function $y = a^x$.

Thus, if $f(x) = a^x$, then $f^{-1}(x) = \log_a x$.

Based on the discussion in Section 5.2, you know that for a function f and its inverse f^{-1},

Domain of f = Range of f^{-1} and Range of f = Domain of f^{-1}.

So, it follows that the Domain of the logarithmic function = Range of the exponential function $= (0, \infty)$

and the Range of the logarithmic function = Domain of the exponential function $= (-\infty, \infty)$.

Summarizing some properties of the logarithmic function:

$y = \log_a x$ (defining equation: $x = a^y$)

Domain: $0 < x < \infty$ Range: $-\infty < y < \infty$

The domain of a logarithmic function consists of the positive real numbers, so the argument of a logarithmic function must be greater than zero.

Example 3: Find the domain of each logarithmic function.

a) $f(x) = \log_4(2 - x)$

Need $2 - x > 0$

$\Rightarrow x < 2$

Domain $= \{x \mid x < 2\}$ or $(-\infty, 2)$

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b) $g(x) = \log_{\frac{1}{3}}(x^2 - 4)$

Need $x^2 - 4 > 0$

$(x+2)(x-2) > 0$

Set $(x+2)(x-2) = 0$

$\Rightarrow x+2=0$ or $x-2=0$

$x=-2$ or $x=2$

	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$	
$x+2$	$-$	$+$	$+$	
$x-2$	$-$	$-$	$+$	
$(x+2)(x-2)$	$+$	$-$	$+$	Want $x^2 - 4 > 0$

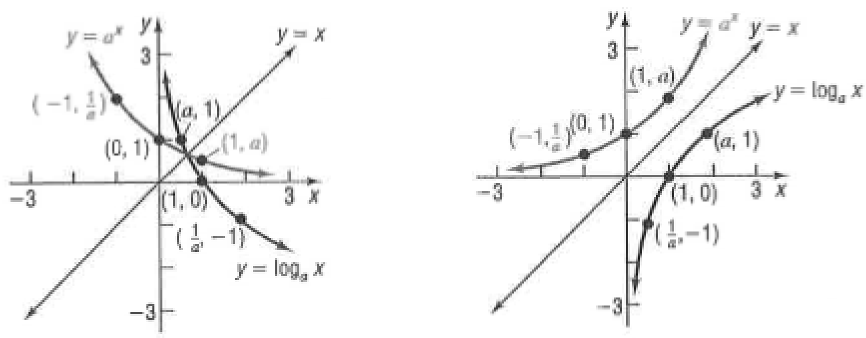
Domain $= \{x \mid x < -2 \text{ or } x > 2\}$ or $(-\infty, -2) \cup (2, \infty)$

c) $h(x) = \log_{\frac{1}{4}}|x|$

Since $|x| > 0$, provided $x \neq 0$, the domain of $h(x)$ consists of all real numbers except zero, or using interval notation, $(-\infty, 0) \cup (0, \infty)$.

Graphs of Logarithmic Functions

Since exponential functions and logarithmic functions are inverses of each other, the graph of the logarithmic function $y = \log_a x$ is the reflection about the line $y = x$ of the graph of the exponential function $y = a^x$, as shown below.



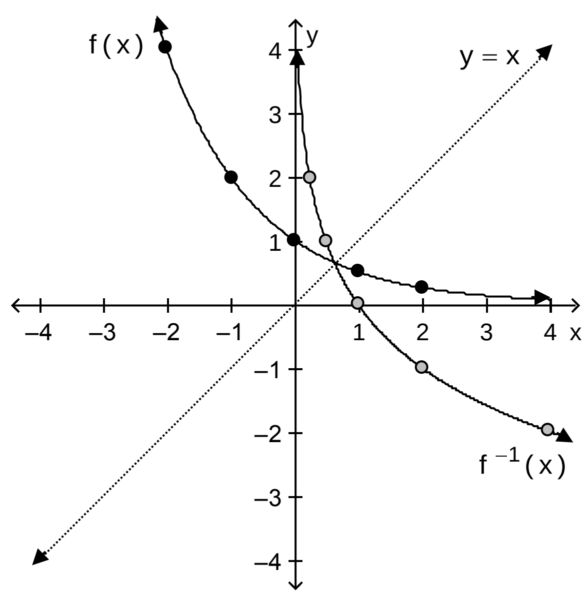
(a) $0 < a < 1$ (b) $a > 1$

If base a is such that $0 < a < 1$:

Example 4: Graph $f(x) = (0.5)^x$ and $f^{-1}(x) = \log_{0.5} x$.

x	$f(x) = (0.5)^x$
−2	4
−1	2
0	1
1	0.5
2	0.25
0.64118574	0.64118574

x	$f^{-1}(x) = \log_{0.5} x$
0.25	2
0.5	1
1	0
2	−1
4	−2
0.64118574	0.64118574



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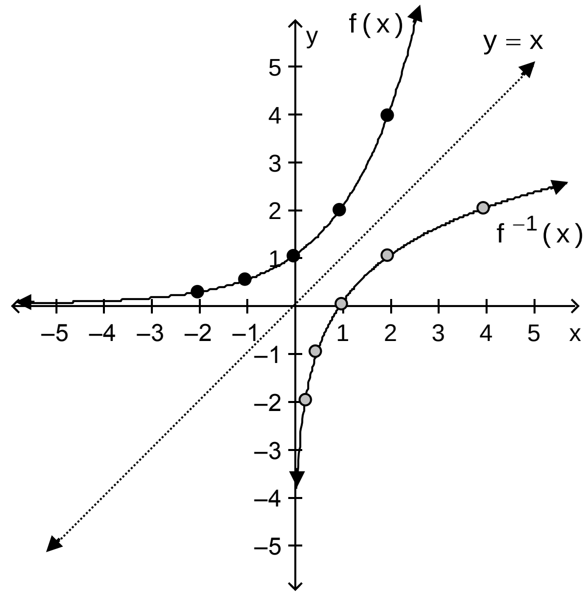
If base a is such that $a > 1$:

Example 5: Graph $f(x) = 2^x$ and $f^{-1}(x) = \log_2 x$.

x	$f(x) = 2^x$
-10	0.000977
-5	0.03125
-2	0.25
-1	0.5
0	1
1	2
2	4
3	8
4	16

x	$f^{-1}(x) = \log_2 x$
0.000977	-10
0.03125	-5
0.25	-2
0.5	-1
1	0
2	1
4	2
8	3
16	4

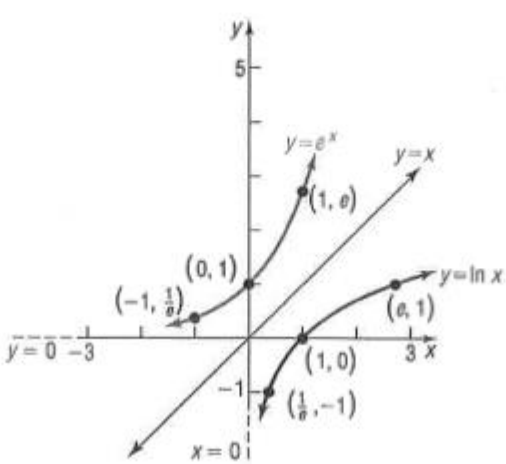


Properties of the Graph of a Logarithmic Function $f(x) = \log_a x$; $a > 0, a \neq 1$

1) The domain is the set of positive real numbers, or $(0, \infty)$ using interval notation; the range is the set of all real numbers, or $(-\infty, \infty)$ using interval notation.
2) The x-intercept of the graph is $(1, 0)$. There is no y-intercept.
3) The y-axis ($x = 0$) is a vertical asymptote of the graph.
4) A logarithmic function is decreasing if $0 < a < 1$ and increasing if $a > 1$.
5) The graph of f contains the points $(1, 0)$, $(a, 1)$, and $\left(\frac{1}{a}, -1\right)$.
6) The graph is smooth and continuous, with no corners or gaps.

If the base of a logarithmic function is the number *e*, the result is the **natural logarithm function**. This function occurs so frequently in applications that it is given a special symbol, $\ln$ (from the Latin, Logarithmus naturalis). That is, $y = \ln x$ if and only if $x = e^y$. (The symbol $\ln$ stands for $\log_e$, so $\ln x \equiv \log_e x$.)

Because $y = \ln x$ and the exponential function $y = e^x$ are inverse functions, the graph of $y = \ln x$ can be obtained by reflecting the graph of $y = e^x$ about the line $y = x$.



Using a calculator with an $\ln$ key, you can obtain other points on the graph of $f(x) = \ln x$. See below.

x	$\ln x$
$\frac{1}{2}$	-0.69
2	0.69
3	1.10

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Example 6: Graph $f(x) = \ln(-x)$.

Start with the basic function $g(x) = \ln x$.

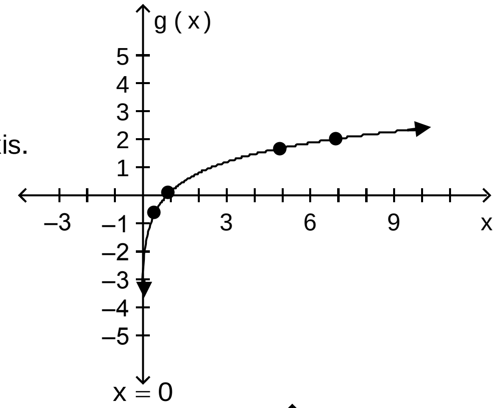
Then $f(x) = g(-x)$ is a reflection of $g(x) = \ln x$ about the y-axis.

$= \ln(-x)$

Domain of $f(x)$: $(-\infty,\ 0)$

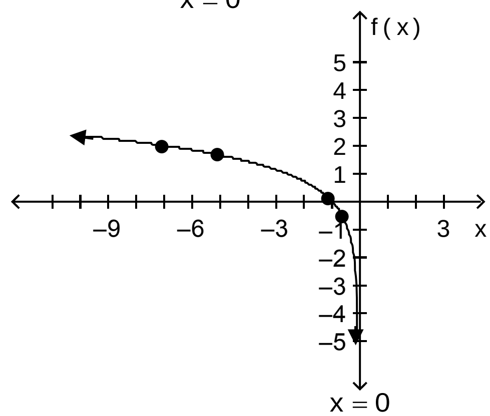
Range of $f(x)$: $(-\infty,\ \infty)$

VA: $x = 0$



x	$g(x) = \ln(x)$
0.01	−4.61
0.1	−2.30
0.5	−0.69
1	0
2	0.69
5	1.61
7	1.95
10	2.30

x	$f(x) = \ln(-x)$
−10	2.30
−7	1.95
−5	1.61
−2	0.69
−1	0
−0.5	−0.69
−0.1	−2.30
−0.01	−4.61



Example 7: Graph $f(x) = \ln(x-4)$.

Start with the basic function $g(x) = \ln x$.

Then $f(x) = g(x-4)$ is a horizontal shift of $g(x) = \ln x$ right 4 units.

$= \ln(x-4)$

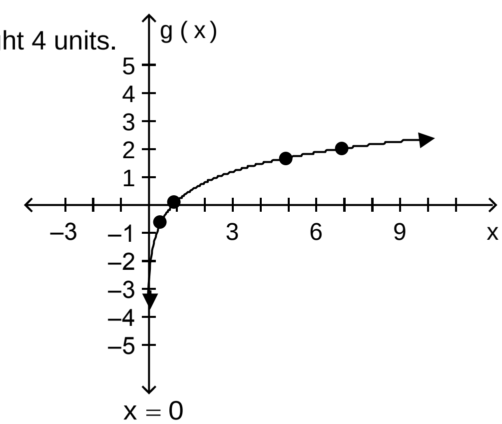
Domain: $x - 4 > 0$

$x > 4$

Domain $= \{x \mid x > 4\}$ or $(4,\ \infty)$

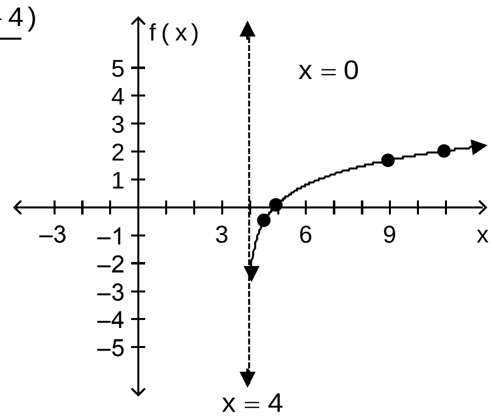
Range: $(-\infty,\ \infty)$

VA: $x = 4$



x	$g(x) = \ln(x)$
0.01	−4.61
0.1	−2.30
0.5	−0.69
1	0
2	0.69
5	1.61
7	1.95
10	2.30

x	$f(x) = \ln(x-4)$
4.01	−4.61
4.1	−2.30
4.5	−0.69
5	0
6	0.69
9	1.61
11	1.95
14	2.30



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You already know that ℓn is the abbreviation for the natural logarithm, or $\log_e$. If the base of a logarithmic function is the number 10 $(\log_{10})$, the result is the **common logarithm function**. If the base *a* of the logarithmic function is not indicated, it is understood to be 10. That is, $y = \log x$ if and only if $x = 10^y$.

Because $y = \log x$ and the exponential function $y = 10^x$ are inverse functions, the graph of $y = \log x$ can be obtained by reflecting the graph of $y = 10^x$ about the line $y = x$.