Section 5.5 – Properties of Logarithms – Day 1

Work with the Properties of Logarithms

Many properties of logarithms can be derived directly from the definition of a logarithm and the laws of exponents. You can easily show that $\log_a 1 = 0$ and $\log_a a = 1$.

Let $y = \log_a 1$. Then $a^y = 1$

$a^y = a^0 \quad a^0 = 1$ since $a > 0$ and $a \neq 1$

$\Rightarrow \quad y = 0$

$\log_a 1 = 0$ since $y = \log_a 1$

Let $y = \log_a a$. Then $a^y = a$

$a^y = a^1$

$\Rightarrow \quad y = 1$

$\log_a a = 1$ since $y = \log_a a$

To summarize: $\log_a 1 = 0$ and $\log_a a = 1$.

Theorem: Properties of Logarithms

M and *a* are positive real numbers, with $a \neq 1$, and r is any real number. Then

1) The number $\log_a M$ is the exponent to which *a* must be raised to obtain M. That is, $a^{\log_a M} = M$.
2) The logarithm with base *a* of *a* raised to a power equals that power. That is $\log_a a^r = r$.

Proof: 1) Let $x = \log_a M$, then $a^x = M$.
But $x = \log_a M$, so $a^{\log_a M} = M$.

2) Let $x = a^r$, then $\log_a x = r$.
But $x = a^r$, so $\log_a a^r = r$.

Example 1: a) $4^{\log_4 7} = 7$ b) $\ln e^{5x} = \log_e e^{5x}$
$= 5x$

Theorem: Properties of Logarithms

In the following properties, M, N, and *a* are positive real numbers, with $a \neq 1$, and r is any real number.

The Log of a Product Equals the Sum of the Logs

$\log_a (MN) = \log_a M + \log_a N$ (Known as the Product Property of Logarithms)

Proof: Let $x = \log_a M$ and $y = \log_a N$.

Then $a^x = M$ and $a^y = N$.

So, $\log_a (MN) = \log_a (a^x a^y)$

$= \log_a a^{x+y}$ by Law of Exponents

$= x + y$ by Property of Logarithms [2) above]

$= \log_a M + \log_a N$

The Log of a Quotient Equals the Difference of the Logs

$\log_a \left(\frac{M}{N} \right) = \log_a M - \log_a N$ (Known as the Quotient Property of Logarithms)

The Log of a Power Equals the Product of the Power and the Log

$\log_a M^r = r \log_a M$ (Known as the Power Property of Logarithms)

Proof: Let $A = \log_a M$, then $a^A = M$.

So, $\log_a M^r = \log_a (a^A)^r$

$= \log_a a^{rA}$ by Law of Exponents

$= r A$ by Property of Logarithms [2) above]

$= r \log_a M$

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We also have $\log_a\left(\frac{1}{N}\right) = -\log_a N$.

Proof: $\log_a\left(\frac{1}{N}\right) = \log_a N^{-1}$ by Law of Exponents

$= -1\log_a N$ by Power Property of Logarithms

$= -\log_a N$

These properties are useful in simplifying problems and solving equations. Pay close attention to the domain when a logarithm's argument contains variables.

Example 2: Simplify

a) $6\log_4(64)$

$$6\log_4(64) = 6\log_4 4^3 = (3)(6)\log_4 4 = (18)(1) = 18$$

b) $4\log_5\left(\frac{1}{125}\right)$

$$4\log_5\left(\frac{1}{125}\right) = 4\log_5 5^{-3} = (-3)(4)\log_5 5 = (-12)(1) = -12$$

c) $\log_2\left[(2)(8)\right]$

$$\log_2\left[(2)(8)\right] = \log_2 2 + \log_2 8 = \log_2 2 + \log_2 2^3 = \log_2 2 + 3\log_2 2 = 1 + 3(1) = 1 + 3 = 4$$

Write a Logarithmic Expression as a Sum or Difference of Logarithms

Logarithms can be used to transform products into sums, quotients into differences, and powers into factors. Such transformations prove useful in certain types of calculus problems.

Example 3: Write $\log_a\left(\frac{x^4}{(x+1)^2}\right)$ as a difference of logarithms.

$$\log_a\left(\frac{x^4}{(x+1)^2}\right) = \log_a x^4 - \log_a(x+1)^2$$

$$= 4\log_a x - 2\log_a(x+1) \quad \text{for } x > 0$$

Example 4: Write $\log_a\left(\frac{(x-3)^4}{x^5\sqrt{x^2+1}}\right)$ $x > 0$

as a sum and difference of logarithms. Express all powers as factors.

$$\log_a\left(\frac{(x-3)^4}{x^5\sqrt{x^2+1}}\right) = \log_a(x-3)^4 - \log_a\left(x^5\sqrt{x^2+1}\right)$$

$$= \log_a(x-3)^4 - \left(\log_a x^5 + \log_a\sqrt{x^2+1}\right)$$

$$= \log_a(x-3)^4 - \log_a x^5 - \log_a\sqrt{x^2+1}$$

$$= \log_a(x-3)^4 - \log_a x^5 - \log_a(x^2+1)^{\frac{1}{2}}$$

$$= 4\log_a(x-3) - 5\log_a x - \frac{1}{2}\log_a(x^2+1)$$

Write a Logarithmic Expression as a Single Logarithm

Another use of the properties is to write sums and/or differences of logarithms with the same base as a single logarithm. This skill will be needed to solve certain logarithmic equations discussed in the next section.

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Example 5: Write as a single logarithm.

a) $\log_2 5 + \log_2 7$

$$\log_2 5 + \log_2 7 = \log_2[(5)(7)]$$
$$= \log_2(35)$$

b) $\log_2(12) - \log_2 6 + \log_2 3$

$$\log_2(12) - \log_2 6 + \log_2 3 = \log_2\left(\frac{12}{6}\right) + \log_2 3$$
$$= \log_2 2 + \log_2 3$$
$$= \log_2[(2)(3)]$$
$$= \log_2 6$$

Students often make the following three errors.

1) Express the logarithm of a sum as the sum of logarithms.

$$\log_a(M+N) \neq \log_a M + \log_a N$$

Correct statement: $\log_a(MN) = \log_a M + \log_a N$

2) Express the difference of logarithms as the quotient of logarithms.

$$\log_a M - \log_a N \neq \frac{\log_a M}{\log_a N}$$

Correct statement: $\log_a M - \log_a N = \log_a\left(\frac{M}{N}\right)$

3) Express a logarithm raised to a power as the product of the power times the logarithm.

$$(\log_a M)^r \neq r\log_a M$$

Correct statement: $\log_a M^r = r\log_a M$

Two other important properties of logarithms are consequences of the fact that the logarithmic function $y = \log_a x$ is a one-to-one function.

Theorem:

In the following properties, M, N, and *a* are positive real numbers, with $a \neq 1$.

If $M = N$, then $\log_a M = \log_a N$.

If $\log_a M = \log_a N$, then $M = N$.

This theorem is useful for solving exponential and logarithmic equations, a topic discussed in the next section.