Section 5.6 – Logarithmic and Exponential Equations

Solve Logarithmic Equations

In Section 5.4, you solved logarithmic equations by changing a logarithmic expression to an exponential expression. That is, you used the definition of a logarithm: $y = \log_a x$ is equivalent to $x = a^y \quad a > 0,\ a \neq 1$.

Example 1: Solve $\log_4(3x+4) = 2$.

Change to exponential form:

$$3x+4 = 4^2$$
$$3x+4 = 16$$
$$3x = 12$$
$$x = 4$$

Check:
$$\log_4\left(3(4)+4\right) = \log_4(16) = \log_4(4^2) = 2\log_4 4 = 2(1) = 2$$

Equations that contain terms of the form $\log_a x$, where *a* is a positive real number, with $a \neq 1$, are called logarithmic equations. You can often solve logarithmic equations by changing the logarithm to exponential form. Sometimes manipulation of the equation is required before you can change it to exponential form. When solving these types of equations, first try to find exact solutions using algebraic methods. When algebraic methods cannot be used, use your graphing calculator to find an approximate solution. When solving logarithmic equations algebraically, be sure to check each of your apparent solutions in the original equation and discard any that are extraneous. Or, to avoid extraneous solutions with logarithmic equations, determine the domain of the variable first.

The next example is of a logarithmic equation that requires using the fact that a logarithmic function is a one-to-one function: If $\log_a M = \log_a N$, then $M = N$ M, N, and *a* are positive and $a \neq 1$.

Example 2: Solve $2\log_6 x = \log_6 25$.

$\log_6 x^2 = \log_6 25$ by the power property

$\Rightarrow x^2 = 25$ (If $\log_a M = \log_a N$, then $M = N$.)

$x = \pm\sqrt{25}$

$x = \pm 5$

Using the original equation to check the solutions, we see we must discard -5, since the logarithm of a negative number is not defined (extraneous solution). Thus, the solution is $x = 5$.

(Or, at the beginning of the problem, state the domain for this equation is $\{x \mid x > 0\}$, therefore -5 is extraneous and must be discarded.)

Example 3: Solve $\log_8(x+3) + \log_8(x-4) = 1$.

$\log_8\left[(x+3)(x-4)\right] = 1$ Write as a single log using the product property

$8^1 = \left[(x+3)(x-4)\right]$ Change to exponential form

$8 = x^2 - x - 12$

$x^2 - x - 20 = 0$

$(x+4)(x-5) = 0$

$\Rightarrow (x+4) = 0$ or $(x-5) = 0$

$x = -4$ or $x = 5$ You can easily check $x = 5$. $x = -4$ is an extraneous solution, since the logarithm of a negative number is not defined. Discard $x = -4$.

Solution: $x = 5$

Be sure to check your solutions in the original equation! Discard any solutions that are extraneous. Remember that in the expression $\log_a M$, a and M are positive and $a \neq 1$.

A negative solution is not automatically extraneous. You must determine whether the potential solution causes the argument of any logarithmic expression in the equation to be negative or 0.

Section 5.6 – Logarithmic and Exponential Equations (continued)

Warning: In using properties of logarithms to solve logarithmic equations, avoid using the property $\log_a x^r = r\log_a x$, when r is even. The reason can be seen in the example below.

Example 4: Solve $\log_2 x^2 = 4$.

The domain of the variable is all real numbers except 0.

a) $\log_2 x^2 = 4$

$\Rightarrow x^2 = 2^4$ change to exponential form

$x^2 = 16$

$x = \pm 4$

b) $\log_2 x^2 = 4$ $\quad \log_a x^r = r\log_a x$

$2\log_2 x = 4$ domain is $x > 0$

$\log_2 x = 2$

$x = 4$

Both -4 and 4 are solutions of $\log_2 x^2 = 4$ (as you can verify). The solution in b), however, does not find the solution -4 because the domain of the variable was further restricted due to the application of the property $\log_a x^r = r\log_a x$.

Solve Exponential Equations

Equations that involve terms of the form a^x, $a > 0$, $a \neq 1$, are referred to as exponential equations.

In Sections 5.3 and 5.4 , you solved exponential equations algebraically by expressing each side of the equation using the same base. One method used to solve exponential equations requires you to rewrite each side of the equation with the same base. Then use the fact that if $a^u = a^v$, then $u = v$.

Example 5: Solve $4^{x-1} = 64$.

$4^{x-1} = 4^3$, same base

$\Rightarrow x - 1 = 3$

$x = 4$

Example 6: Solve $9^x - 3^x - 72 = 0$.

$(3^2)^x - 3^x - 72 = 0$

$(3^x)^2 - 3^x - 72 = 0$

$(3^x - 9)(3^x + 8) = 0$

$\Rightarrow (3^x - 9) = 0$ or $(3^x + 8) = 0$

$3^x = 9$ $\qquad 3^x = -8$

$3^x = 3^2$, same base $\qquad$ But $3^x > 0\ \forall x$,

$\Rightarrow x = 2$ $\qquad$ so no real solution.

Solution: $x = 2$

Many exponential equations, however, cannot be rewritten so that each side has the same base. In such cases, using properties of logarithms, along with algebraic techniques, can sometimes be used to obtain a solution.

Example 7: Solve $3^x = 6$.

There are two methods of solving this equation.

a) Write the equivalent log equation:

$x = \log_3 6$

$= \dfrac{\log 6}{\log 3}$ or $\dfrac{\ln 6}{\ln 3}$

≈ 1.63

b) $3^x = 6$

$\Rightarrow \log 3^x = \log 6$

$x \log 3 = \log 6$

$x = \dfrac{\log 6}{\log 3}$

≈ 1.63

Section 5.6 – Logarithmic and Exponential Equations (continued)

Example 8: Solve $3^{x+4} = 7^{2x+1}$.

$$\ln\left(3^{x+4}\right) = \ln\left(7^{2x+1}\right)$$
$$(x+4)\ln 3 = (2x+1)\ln 7$$
$$x\ln 3 + 4\ln 3 = 2x\ln 7 + \ln 7$$
$$x\ln 3 - 2x\ln 7 = \ln 7 - 4\ln 3$$
$$x(\ln 3 - 2\ln 7) = \ln 7 - 4\ln 3$$
$$x = \frac{\ln 7 - 4\ln 3}{\ln 3 - 2\ln 7} \quad \ln \text{ form}$$
$$x \approx 0.876605$$
$$x \approx 0.88 \quad \text{decimal form}$$

Example 9: Solve $e^{-2x+1} = 13$.

Change to logarithmic form:

$$\ln(13) = -2x + 1$$
$$\ln(13) - 1 = -2x$$
$$x = \frac{\ln(13) - 1}{-2} \quad \ln \text{ form}$$
$$x \approx -0.782475 \quad \text{decimal form}$$

Check: $e^{-2\left(\frac{\ln(13)-1}{-2}\right)+1}$
$$= e^{(\ln(13)-1)+1}$$
$$= e^{\ln(13)}$$
$$= 13$$

Solve Logarithmic and Exponential Equations Using a Graphing Utility

The algebraic techniques in this section to obtain exact solutions apply only to certain types of logarithmic and exponential equations. Solutions for other types are generally studied in calculus, using numerical methods. For such types, you can use a graphing utility to approximate the solution.

So, If you cannot get a solution using algebraic methods, use your calculator.

Example 10: Solve $\log_5 x + \log_7 x = 2$ to two decimal places algebraically and using your calculator.

a) Algebraically:

$$\log_5 x + \log_7 x = 2$$
$$\frac{\log x}{\log 5} + \frac{\log x}{\log 7} = 2$$
$$\log 5 \log 7\left(\frac{\log x}{\log 5} + \frac{\log x}{\log 7}\right) = 2\log 5 \log 7$$
$$\log 7 \log x + \log 5 \log x = 2\log 5 \log 7$$
$$(\log 7 + \log 5)\log x = 2\log 5 \log 7$$
$$\log x = \frac{2\log 5 \log 7}{\log 7 + \log 5}$$
$$\Rightarrow 10^{\frac{2\log 5 \log 7}{\log 7 + \log 5}} = x \quad \text{Change to exponential form}$$
$$10^{0.765119} \approx x$$
$$x \approx 5.822627$$
$$x \approx 5.82$$

Example 10: (continued)

Solve $\log_5 x + \log_7 x = 2$ to two decimal places using your calculator.

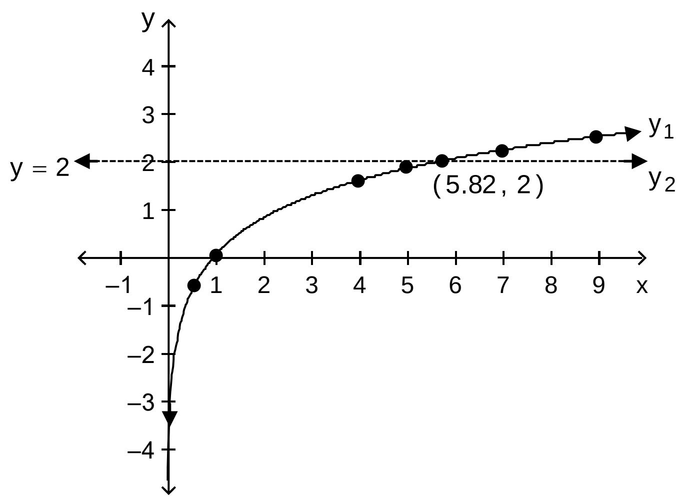
b) Using your Calculator:

Graph $y_1 = \log_5 x + \log_7 x$ and $y_2 = 2$.

$$= \frac{\log x}{\log 5} + \frac{\log x}{\log 7}$$

Find their intersection point.

y_1 is an increasing function, so there is only one point of intersection.



x	$y_1 = \log_5 x + \log_7 x$
0.5	–0.786884
1	0
2	0.786884
3	1.247181
4	1.573768
5	1.827088
6	2.034065
7	2.209062
9	2.494363

Intersection point: $\approx (5.822632\ ,\ 2)$

$\approx (5.82,\ 2)$

All material has been taken from *Precalculus,* by M. Sullivan, 10th Edition