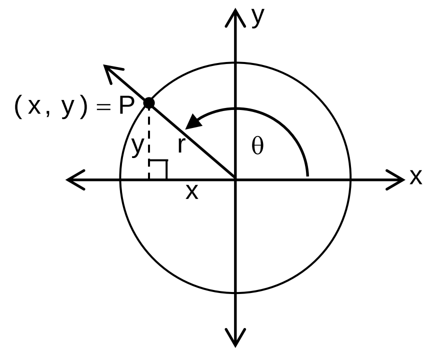
Section 6.2 – Trigonometric Functions: Unit Circle Approach – Day 1

Recall that the unit circle is a circle whose radius is 1 and whose center is at the origin of a rectangular coordinate system. Also recall that any circle of radius *r* has circumference of length $2\pi r$. Therefore, the unit circle (radius = 1) has a circumference of length 2π. So, for 1 revolution around the unit circle, the length of the arc is 2π units.

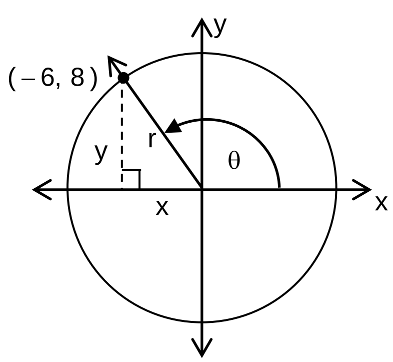
Theorem:
For an angle θ in standard position, let $P = (x, y)$ be the point on the terminal side of θ that is also on the circle $x^2 + y^2 = r^2$.

Then $\sin\theta = \frac{y}{r}$ $\cos\theta = \frac{x}{r}$ $\tan\theta = \frac{y}{x},\ x \neq 0$

$\csc\theta = \frac{r}{y},\ y \neq 0$ $\sec\theta = \frac{r}{x},\ x \neq 0$ $\cot\theta = \frac{x}{y},\ y \neq 0$



Example 1: Find the exact value of each of the six trigonometric functions of an angle θ if $(-6, 8)$ is a point on its terminal side.



$(x, y) = (-6, 8)$
$\Rightarrow x = -6,\ y = 8$

$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ Distance formula

$r = \sqrt{(x - 0)^2 + (y - 0)^2}$ Distance from the origin, $(0, 0)$, to the point (x, y)

$$r = \sqrt{x^2 + y^2} = \sqrt{(-6)^2 + 8^2} = \sqrt{36 + 64} = \sqrt{100} = 10$$

$\Rightarrow (-6, 8)$ is a point on the circle $x^2 + y^2 = r^2$

$$x^2 + y^2 = 10^2$$
$$x^2 + y^2 = 100$$

$$\sin\theta = \frac{y}{r} = \frac{8}{10} = \frac{4}{5} \qquad \cos\theta = \frac{x}{r} = \frac{-6}{10} = \frac{-3}{5} \qquad \csc\theta = \frac{r}{y} = \frac{10}{8} = \frac{5}{4}$$

$$\sec\theta = \frac{r}{x} = \frac{10}{-6} = \frac{-5}{3} \qquad \tan\theta = \frac{y}{x} = \frac{8}{-6} = \frac{-4}{3} \qquad \cot\theta = \frac{x}{y} = \frac{-6}{8} = \frac{-3}{4}$$

If an angle θ is measured in degrees, use the degree symbol when writing a trigonometric function of θ, such as $\sin 30^\circ$. If an angle θ is measured in radians, then no symbol is used when writing a trigonometric function of θ, such as $\tan\pi$.

Section 6.2 – Trigonometric Functions: Unit Circle Approach – Day 1 (continued)

Recall that if the terminal side of an angle θ lies on the x- or y-axis, then we say θ is a **quadrantal** angle.
The four basic quadrantal angles are $0 = 0^\circ$, $\frac{\pi}{2} = 90^\circ$, $\pi = 180^\circ$, and $\frac{3\pi}{2} = 270^\circ$.

Example 2: Find the exact value of the six trigonometric functions at $\theta = \frac{3\pi}{2}$.

The point P = $(0, -1)$ is on the terminal side of $\theta = \frac{3\pi}{2}$ and is on the unit circle.

$$P = (x, y) = (0, -1) \Rightarrow x = 0,\ y = -1$$

$$r = \sqrt{x^2 + y^2} = \sqrt{(0)^2 + (-1)^2} = \sqrt{0+1} = \sqrt{1} = 1$$

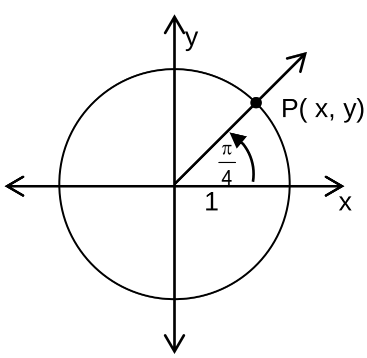
$$\sin\theta = \frac{y}{r} = \frac{-1}{1} = -1 \qquad \cos\theta = \frac{x}{r} = \frac{0}{1} = 0 \qquad \csc\theta = \frac{r}{y} = \frac{1}{-1} = -1$$

$$\sec\theta = \frac{r}{x} = \frac{1}{0} \text{ undefined} \qquad \tan\theta = \frac{y}{x} = \frac{-1}{0} \text{ undefined} \qquad \cot\theta = \frac{x}{y} = \frac{0}{-1} = 0$$

Example 3: Find the exact value of the six trigonometric functions of $\frac{\pi}{4}$.

The point $P = (x, y)$ is on the terminal side of $\theta = \frac{\pi}{4}$. P lies on the line $y = x$, so x = y.

Suppose $P = (x, y)$ lies on the unit circle, then $x^2 + y^2 = 1$. But x = y, thus $x^2 + x^2 = 1$.



$$2x^2 = 1$$

$$x^2 = \frac{1}{2}$$

P in quadrant I, so take the positive square root $\quad x = \frac{1}{\sqrt{2}}$

$$x = \frac{\sqrt{2}}{2}$$

$$x = y \Rightarrow y = \frac{\sqrt{2}}{2} \text{ too}$$

$$\sin\frac{\pi}{4} = \frac{y}{r} = \frac{\sqrt{2}/2}{1} = \frac{\sqrt{2}}{2}$$

$$\cos\frac{\pi}{4} = \frac{x}{r} = \frac{\sqrt{2}/2}{1} = \frac{\sqrt{2}}{2}$$

$$\csc\frac{\pi}{4} = \frac{r}{y} = \frac{1}{\sqrt{2}/2} = \frac{2}{\sqrt{2}} = \frac{2}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

$$\sec\frac{\pi}{4} = \frac{r}{x} = \frac{1}{\sqrt{2}/2} = \frac{2}{\sqrt{2}} = \frac{2}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

$$\tan\frac{\pi}{4} = \frac{y}{x} = \frac{\sqrt{2}/2}{\sqrt{2}/2} = 1$$

$$\cot\frac{\pi}{4} = \frac{x}{y} = \frac{\sqrt{2}/2}{\sqrt{2}/2} = 1$$

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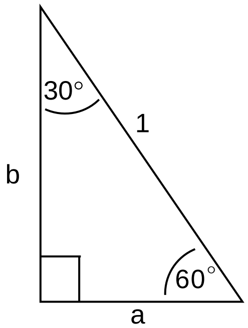
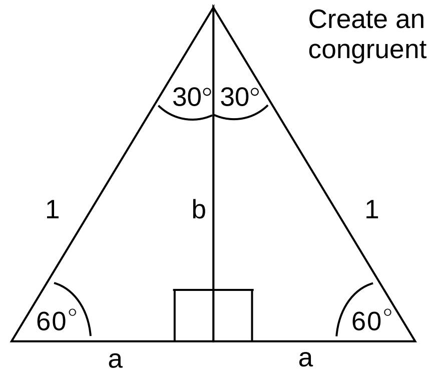
Consider a 30° - 60° - 90° triangle with a hypotenuse of 1. Find the lengths of sides a and b.





Create an equilateral triangle by placing a congruent triangle next to the first triangle.

base $2a = 1$

$a = \frac{1}{2}$

Also, $a^2 + b^2 = 1$

$\left(\frac{1}{2}\right)^2 + b^2 = 1$

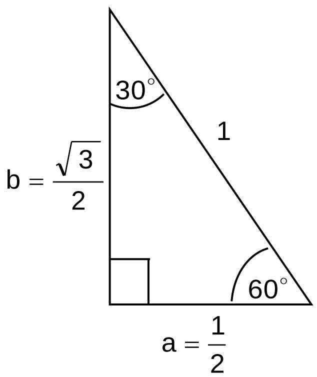
$\frac{1}{4} + b^2 = 1$

$b^2 = \frac{3}{4}$

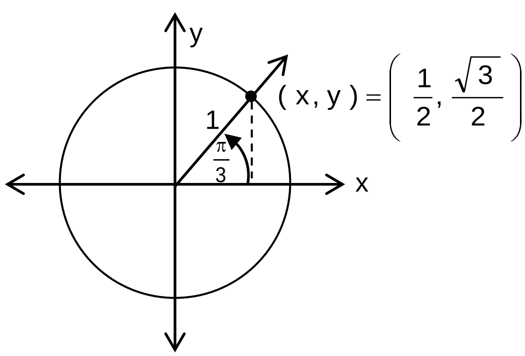
$b = \frac{\sqrt{3}}{2}$

Take the positive square root, since b represents a length.

Thus, we have



Example 4: Find the exact value of the six trigonometric functions of $\frac{\pi}{3}$.



$\sin\frac{\pi}{3} = \frac{y}{r} = \frac{\sqrt{3}/2}{1} = \frac{\sqrt{3}}{2}$

$\cos\frac{\pi}{3} = \frac{x}{r} = \frac{1/2}{1} = \frac{1}{2}$

$\csc\frac{\pi}{3} = \frac{r}{y} = \frac{1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3}}\right) = \frac{2\sqrt{3}}{3}$

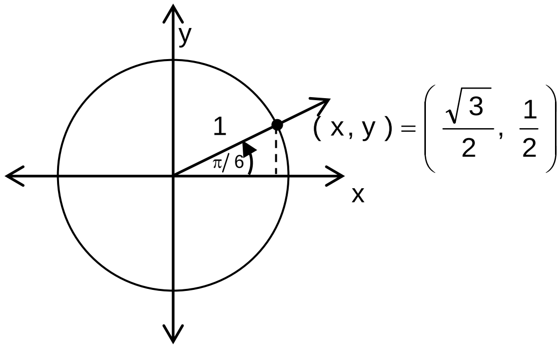
$\sec\frac{\pi}{3} = \frac{r}{x} = \frac{1}{1/2} = 2$

$\tan\frac{\pi}{3} = \frac{y}{x} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$

$\cot\frac{\pi}{3} = \frac{x}{y} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3}}\right) = \frac{\sqrt{3}}{3}$

Section 6.2 – Trigonometric Functions: Unit Circle Approach – Day 1 (continued)

Example 5: Find the exact value of the six trigonometric functions of $\frac{\pi}{6}$.



$\sin\frac{\pi}{6} = \frac{y}{r} = \frac{1/2}{1} = \frac{1}{2}$

$\cos\frac{\pi}{6} = \frac{x}{r} = \frac{\sqrt{3}/2}{1} = \frac{\sqrt{3}}{2}$

$\csc\frac{\pi}{6} = \frac{r}{y} = \frac{1}{1/2} = 2$

$\sec\frac{\pi}{6} = \frac{r}{x} = \frac{1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3}}\right) = \frac{2\sqrt{3}}{3}$

$\tan\frac{\pi}{6} = \frac{y}{x} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3}}\right) = \frac{\sqrt{3}}{3}$

$\cot\frac{\pi}{6} = \frac{x}{y} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$

Example 6: Find the exact value of the following:

a) $3\csc\frac{\pi}{3} + \cot\frac{\pi}{4}$

For $\frac{\pi}{3}$, $(x, y) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

For $\frac{\pi}{4}$, $(x, y) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

So, $3\csc\frac{\pi}{3} + \cot\frac{\pi}{4} = 3\left(\frac{r}{y}\right) + \frac{x}{y}$

$= 3\left(\frac{1}{\sqrt{3}/2}\right) + \frac{\sqrt{2}/2}{\sqrt{2}/2}$

$= 3\left(\frac{2}{\sqrt{3}}\right) + 1$

$= 3\left(\frac{2}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{\sqrt{3}}\right) + 1$

$= \cancel{3}\left(\frac{2\sqrt{3}}{\cancel{3}}\right) + 1$

$= 2\sqrt{3} + 1$

b) $\sin\frac{3\pi}{2} + 5\tan\pi$

For $\frac{3\pi}{2}$, $(x, y) = (0, -1)$

For π, $(x, y) = (-1, 0)$

So, $\sin\frac{3\pi}{2} + 5\tan\pi = \frac{y}{r} + 5\left(\frac{y}{x}\right)$

$= \frac{-1}{1} + 5\left(\frac{0}{-1}\right)$

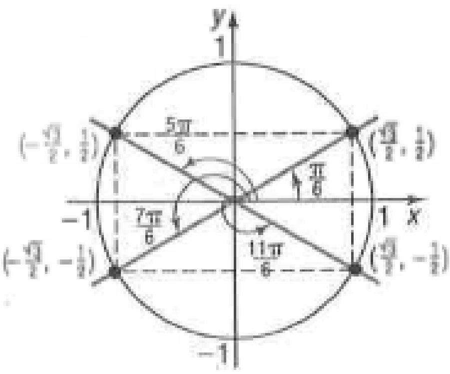
$= -1 + 5(0)$

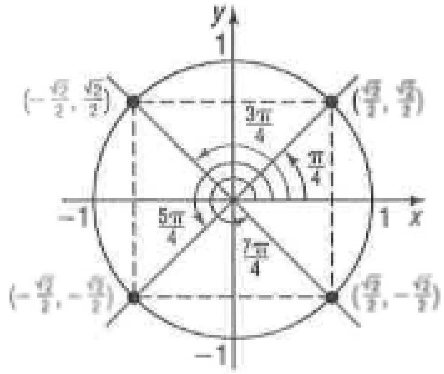
$= -1 + 0$

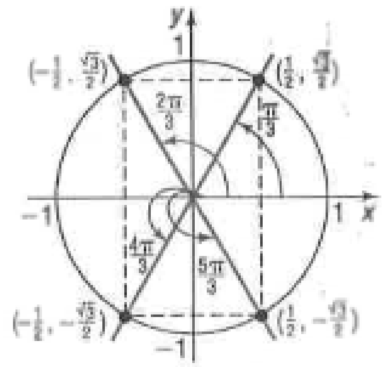
$= -1$

Section 6.2 – Trigonometric Functions: Unit Circle Approach – Day 1

The following figures will help you find the exact values of the trigonometric functions for integral multiples of $\frac{\pi}{6}=30^\circ,\ \frac{\pi}{4}=45^\circ,$ and $\frac{\pi}{3}=60^\circ$.







All material has been taken from ***<u>Precalculus,</u>*** by M. Sullivan, 10th Edition