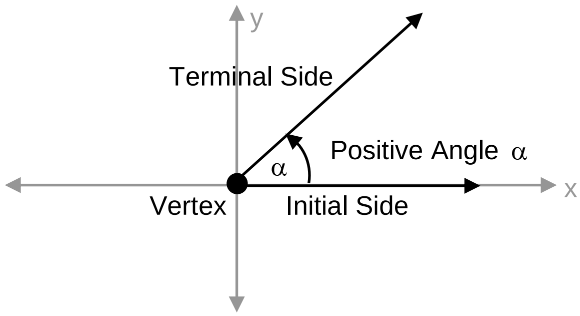
Section 6.1 – Angles and Their Measure

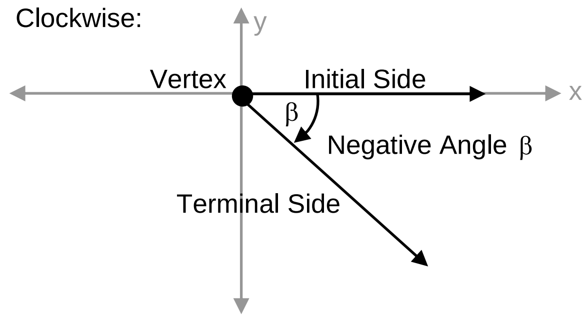
A ray, or half-line, is that portion of a line that starts at a point V on the line and extends infinitely in one direction. The starting point V of a ray is called its vertex.

Two rays that are drawn with a common vertex form an **angle**. We call one ray of an angle the **initial side** and the other the **terminal side**. The angle formed is identified by showing the direction and amount of rotation from the initial side to the terminal side. If the rotation is in the counterclockwise direction, the angle is **positive**; if the rotation is clockwise, the angle is **negative**. Lowercase Greek letters, such as α (alpha), β (beta), γ (gamma), and θ (theta) will be used to denote angles.

Counterclockwise:

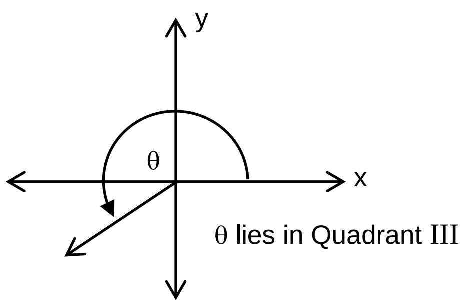


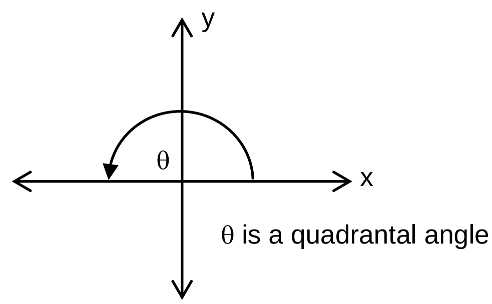
Clockwise:



An angle θ is said to be in **standard position** if its vertex is at the origin of a rectangular coordinate system and its initial side coincides with the positive x-axis. Above, α and β are in standard position.

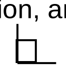
When an angle θ is in standard position, the terminal side will lie either in a quadrant, in which case we say θ **lies in that quadrant**, or the terminal side will lie on the x- or y-axis, in which case we say θ is a **quadrantal angle**.





Angles are measured by determining the amount of rotation needed for the initial side to become coincident with the terminal side. The two commonly used measures for angles are **degrees** and **radians**.

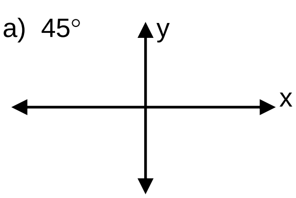
Degrees

The angle formed by rotating the initial side exactly once in the counterclockwise direction until it coincides with itself (1 revolution) is said to measure 360 degrees, written 360°. **1 degree** = 1° is $\frac{1}{360}$ revolution. A **right angle** = 90° is $\frac{1}{4}$ revolution, and a straight angle = 180° is $\frac{1}{2}$ revolution. It is customary to indicate a right angle by using the symbol  .

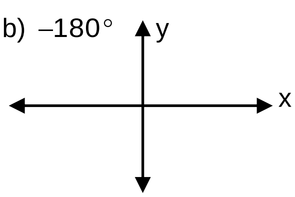
It is customary to refer to an angle that measures θ degrees as an angle *of* θ degrees.

Example 1: Draw angles of the following measures:

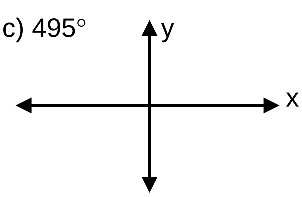
a) 45°



b) –180°



c) 495°



Sec 6.1 – Angles and Their Measure (continued)

Convert between Decimals and Degrees, Minutes, Seconds Measures for Angles

Instead of using decimals for subdivisions of a degree, we also use the terms minute and seconds,

One minute, denoted by $1'$, is defined as $\frac{1}{60}$ degree.

One second, denoted $1''$, is defined as $\frac{1}{60}$ minute $= \frac{1}{3600}$ degree.

So, $1^\circ = 60' \Rightarrow$ 1 degree is 60 minutes
$1' = 60'' \Rightarrow$ 1 minute is 60 seconds

$35^\circ\ 10'\ 20''$ is an angle of 35 degrees, 10 minutes, 20 seconds. It is said to be written in $D^\circ\ M'\ S''$ (degrees, minutes, seconds) form. It is sometimes necessary to convert from $D^\circ\ M'\ S''$ form to a decimal form and vice versa.

Example 2: a) Convert $40^\circ\ 7'\ 16''$ to a decimal in degrees. Round the answer to two decimal places.

Know $1' = \frac{1}{60}^\circ$ and $1'' = \frac{1}{60}'$
$= \frac{1}{3600}^\circ$

$$40^\circ\ 7'\ 16'' = \left(40 + 7\left(\frac{1}{60}\right) + 16\left(\frac{1}{3600}\right)\right)^\circ$$
$$\approx (40 + 0.116667 + 0.004444)^\circ$$
$$\approx 40.121111^\circ$$
$$\approx 40.12^\circ$$

b) Convert 32.345° to the $D^\circ\ M'\ S''$ form.

Change 0.345° to minutes: $0.345^\circ = 0.345(1^\circ)$
$= 0.345(60')$
$= 20.7'$

Change $0.7'$ to seconds: $0.7' = 0.7(1')$
$= 0.7(60'')$
$= 42''$

$$\Rightarrow 32.345^\circ = 32^\circ + 20.7'$$
$$= 32^\circ + 20' + 42''$$
$$= 32^\circ\ 20'\ 42''$$

In many applications, such as finding the precise location of a ship at sea or the exact location of a star, angles measured in degrees, minutes, and seconds ($D^\circ\ M'\ S''$ form) are used. For calculation purposes, these are transformed to decimal form. In other applications, especially those in calculus, angles are measured in radians.

Sec 6.1 – Angles and Their Measure (continued)

Radians

A **central angle** is a positive angle whose vertex is at the center of a circle. The rays of a central angle subtend (intersect) an arc on the circle. If the radius of the circle is *r* and the length of the arc subtended by the central angle is also *r*, then the measure of the angle is **1 radian**.

For a circle of radius 1, the rays of a central angle with measure 1 radian subtend an arc of length 1. See the left figure below. For a circle of radius 3, the rays of a central angle with measure 1 radian subtend and arc of length 3. See Figure 10a below.

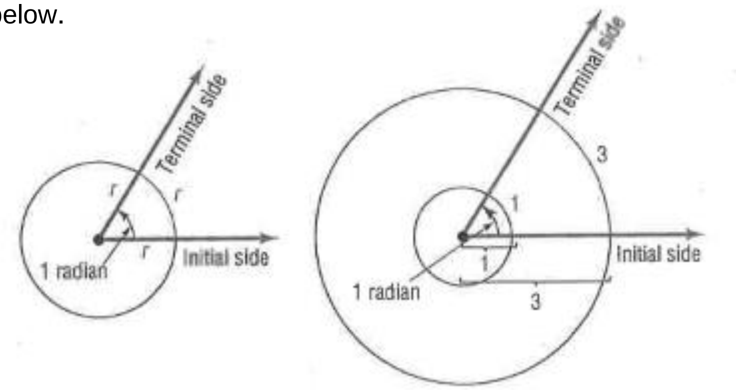


Figure 10 (a) (b)

Find the Length of an Arc of a Circle

Now consider a circle of radius r and two central angles, θ and θ_1, measured in radians. Suppose that these central angles subtend arcs of lengths s and s_1, respectively, as shown in the figure below. From geometry, the ratio of the measures of the angles equals the ratio of the corresponding lengths of the arcs subtended by these angles; that is, $\frac{\theta}{\theta_1} = \frac{s}{s_1}$.

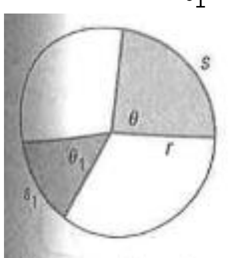
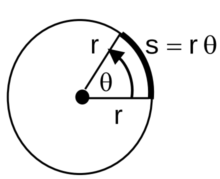


Figure 11 $\frac{\theta}{\theta_1} = \frac{s}{s_1}$

Suppose that $\theta_1 = 1$ radian. The length s_1 of the arc subtended by the central angle $\theta_1 = 1$ radian equals the radius *r* of the circle. (See Fig. 10a above). Then $s_1 = r$, so the equation $\frac{\theta}{\theta_1} = \frac{s}{s_1}$ reduces to $\frac{\theta}{1} = \frac{s}{r}$ or $s = r\theta$.

Arc Length Theorem:
For a circle of radius r, a central angle of θ radians subtends an arc whose length s is given by $s = r\theta$.



*** In using the formula $s = r\theta$, the dimension for θ is radians, and any unit of length may be used for r and s.

Example 3: Find the length of the arc of a circle with radius 3 inches subtended by a central angle of 0.5 radians.

Given: $r = 3$ in. and $\theta = 0.5$ radians

Arc length $s = r\theta$

$= 3(0.5)$

$= 1.5$ inches

Sec 6.1 – Angles and Their Measure (continued)

Convert from Degrees to Radians and from Radians to Degrees

With two ways to measure angles, it is important to be able to convert from one to the other. Consider a circle or radius r. A central angle of 1 revolution will subtend an arc equal to the circumference of the circle (Figure 12). Because the circumference of a circle of radius r equals $2\pi r$, substitute s in $s = r\theta$ to find that, for an angle θ of one revolution,

$$s = r\theta$$
$$2\pi r = r\theta$$
$$\theta = 2\pi \text{ radians}$$

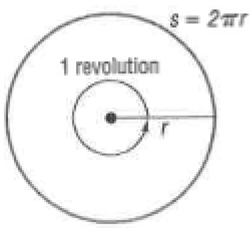


Figure 12
1 revolution = 2π radians

From this, you have 1 revolution $= 2\pi$ radians

Since 1 revolution $= 360^\circ$, you have $360^\circ = 2\pi$ radians

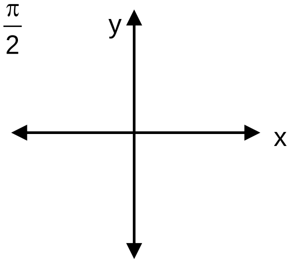
Dividing both sides by 2 gives you $180^\circ = \pi$ radians

So, you get the following two conversion formulas:

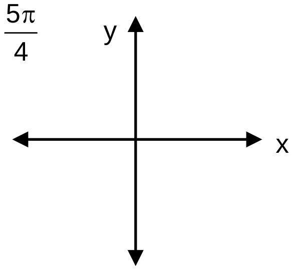
$$1 \text{ degree} = \frac{\pi}{180} \text{ radians} \qquad 1 \text{ radian} = \frac{180}{\pi} \text{ degrees}$$

Example 4: Draw angles of the following measures:

a) $\frac{\pi}{2}$



b) $\frac{5\pi}{4}$



Example 5: Convert each angle in degrees to radians.

a) $80^\circ = 80\,(1 \text{ degree})$

$= 80\left(\frac{\pi}{180} \text{ radians}\right)$

$= \frac{4}{9}\pi$ radians

b) $-120^\circ = -120\,(1 \text{ degree})$

$= -120\left(\frac{\pi}{180} \text{ radians}\right)$

$= \frac{-2}{3}\pi$ radians

Example 6: Convert each angle in radians to degrees.

a) $\frac{\pi}{4}$ radians $= \frac{\pi}{4}(1 \text{ radian})$

$= \frac{\pi}{4}\left(\frac{180}{\pi} \text{ degrees}\right)$

$= 45$ degrees

$= 45^\circ$

b) $\frac{-4}{5}\pi$ radians $= \frac{-4}{5}\pi\,(1 \text{ radian})$

$= \frac{-4}{5}\pi\left(\frac{180}{\pi} \text{ degrees}\right)$

$= \frac{-720}{5}$ degrees

$= -144^\circ$

When an angle is measured in degrees, the degree symbol will be shown. When an angle is measured in radians, the word radians is usually omitted. So, the measure of an angle without units is understood to be in radians.

Table 1 lists the degree and radian measures of commonly encountered angles. You should be comfortable using either measure for these angles.

Table 1

Degrees	0°	30°	45°	60°	90°	120°	135°	150°	180°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
Degrees		210°	225°	240°	270°	300°	315°	330°	360°
Radians		$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π

Sec 6.1 – Angles and Their Measure (continued)

Example 7: Find the length of the arc of a circle with radius 6 inches subtended by a central angle of 25 degrees. Round the answer to two decimal places.

$r = 6\text{ in.} \qquad \theta = 25^\circ$ Need to convert θ to radians

$$\theta = 25\left(\frac{\pi}{180}\right)$$
$$\approx 0.436332 \text{radians}$$

Arc length $s = r\,\theta$

$$= 6(0.436332)$$
$$= 2.617992$$
$$\approx 2.62 \text{ inches}$$

Find the Area of a Sector of a Circle

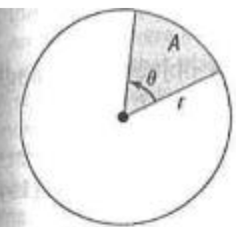


Figure 14 Sector of a Circle

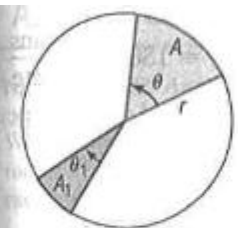


Figure 15 $\dfrac{\theta}{\theta_1} = \dfrac{A}{A_1}$

Consider a circle of radius r. Suppose that θ, measured in radians, is a central angle of this Circle. See Figure 14. We seek a formula for the area A of the sector (the shaded region) formed by angle θ.

Now consider a circle of radius r and two central angles θ and θ_1, both measured in radians. See Figure 15. From geometry, you know that the ratio of the measures of the angles equals the ratio of the corresponding areas of the sectors formed by these angles. That is, $\dfrac{\theta}{\theta_1} = \dfrac{A}{A_1}$.

Suppose that $\theta_1 = 2\pi$ radians. Then $A_1 = \text{area of the circle} = \pi r^2$. Solving for A, you find

$$A = A_1 \frac{\theta}{\theta_1}$$
$$= \pi r^2 \frac{\theta}{2\pi}$$
$$= \frac{1}{2} r^2 \theta$$

Theorem: Area of a Sector

The area A of the sector of a circle of radius r formed by a central angle of θ **radians** is $A = \dfrac{1}{2} r^2 \theta$.

Example 8: Find the area of the sector of a circle of radius 2 feet formed by an angle 60°. Round the answer to two decimal places.

$r = 2$ feet and $\theta = 60^\circ$

You need to convert θ to radians

$$\theta = 60\left(\frac{\pi}{180}\right)$$
$$\theta = \frac{\pi}{3} \text{ radians}$$

The area of the sector is $A = \dfrac{1}{2} r^2 \theta$

$$A = \frac{1}{2}(2)^2 \frac{\pi}{3}$$
$$= \frac{4}{2}\left(\frac{\pi}{3}\right)$$
$$= \frac{2\pi}{3}$$
$$\approx 2.094$$

So, the area of the sector is 2.09 square feet, rounded to two decimal places.

Sec 6.1 – Angles and Their Measure (continued)

Find the Linear Speed of an Object Traveling in Circular Motion

Earlier we defined the average speed of an object as the distance traveled divided by the elapsed time. For motion along a circle, we distinguish between *linear speed* and *angular speed*.

Circular Motion

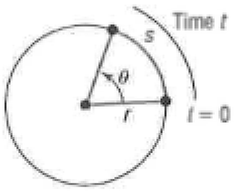


Figure 16 $v = \frac{s}{t}$

Definition: Suppose that an object moves around a circle of radius *r* at a constant speed. If *s* is the distance traveled in time *t* around this circle, then the **linear speed v** of the object is defined as $v = \frac{s}{t}$ (unit dimension is length per unit of time).

As the object travels around the circle, suppose that θ (measured in radians) is the central angle swept out in time t. See Figure 16.

Definition: Then the **angular speed ω** of this object is the angle θ (measured in radians) swept out, divided by the elapsed time t; that is $\omega = \frac{\theta}{t}$ (unit dimension is radians per unit of time).

There is an important relationship between linear speed v and angular speed ω

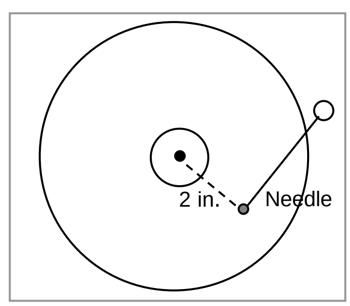
$$\text{Linear speed } v = \frac{s}{t}$$
$$= \frac{r\theta}{t} \quad \text{since} \quad s = r\theta$$
$$= r\omega \quad \text{since} \quad \omega = \frac{\theta}{t}$$

So, $v = r\omega$ where angular speed ω is measured radians per unit of time

When using $v = r\omega$, remember that $v = \frac{s}{t}$ (the linear speed) has the dimensions of length per unit of time, r (the radius of the circular motion) has the same length dimension as s, and ω (the angular speed) has the dimensions of radians per unit of time. If the angular speed is given in terms of *revolutions* per unit of time, be sure to convert it to *radians* per unit of time using the fact that 1 revolution $= 2\pi$ radians.

Example 9: Find the linear speed of a 45 rpm record (rev/min $\Rightarrow$ angular speed) at the point where the needle is 2 inches from the center of the record.

$$\text{Angular speed } \omega = 45 \text{ rpm}$$
$$= 45\,\frac{\text{revolutions}}{\text{minute}}$$
$$= \left(45\,\frac{\text{revolutions}}{\text{minute}}\right)\left(\frac{2\pi \text{ radians}}{\text{revolution}}\right)$$
$$= \frac{90\pi \text{ radians}}{\text{minute}}$$



$$\text{Linear speed } v = r\omega$$
$$= 2 \text{ inches}\left(\frac{90\pi \text{ radians}}{\text{minute}}\right)$$
$$= \frac{180\pi \text{ inches}}{\text{minute}}$$
$$\approx 565.486678 \text{ in/min}$$
$$\approx 565.49 \text{ in/min}$$

All material has been taken from *Precalculus,* by M. Sullivan, 10th Edition