Section 6.3 – Properties of the Trigonometric Functions – Day 1

Domain and Range of the Trigonometric Functions

Let θ be an angle in standard position, and let $P = (x, y)$ be the point on the unit circle that corresponds to θ. Then, by the definition given earlier,

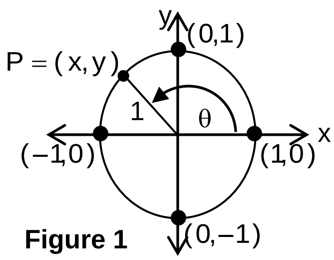

Figure 1

$\sin\theta = y \qquad \cos\theta = x \qquad \tan\theta = \dfrac{y}{x},\ x \neq 0$

$\csc\theta = \dfrac{1}{y},\ y \neq 0 \qquad \sec\theta = \dfrac{1}{x},\ x \neq 0 \qquad \cot\theta = \dfrac{x}{y},\ y \neq 0$

For $\sin\theta$ and $\cos\theta$, there is no concern about dividing by 0, so θ can be any angle. It follows that the domain of the sine function and cosine function is the set of all real numbers.

The domain of the sine function is the set of all real numbers.
The domain of the cosine function is the set of all real numbers.

For the tangent function and secant function, the x-coordinate of $P = (x, y)$ cannot be 0. On the unit circle, there are two such points with $x = 0$, $(0, 1)$ and $(0, -1)$. These two points correspond to the angles $\frac{\pi}{2}\,(90^\circ)$ and $\frac{3\pi}{2}\,(270^\circ)$ or, more generally, to any angle that is an odd integral multiple of $\frac{\pi}{2}\,(90^\circ)$, such as $\pm\frac{\pi}{2}\,(90^\circ)$, $\pm\frac{3\pi}{2}\,(270^\circ)$, $\pm\frac{5\pi}{2}\,(450^\circ)$, and so on. Such angles must be excluded from the domains of the tangent and secant functions.

The domain of the tangent function is the set of all real numbers, except odd integral multiples of $\frac{\pi}{2}\,(90^\circ)$.

The domain of the secant function is the set of all real numbers, except odd integral multiples of $\frac{\pi}{2}\,(90^\circ)$.

For the cotangent function and cosecant function, the y-coordinate of $P = (x, y)$ cannot be 0 since this results in division by 0. On the unit circle, there are two such points with $y = 0$, $(1, 0)$ and $(-1, 0)$. These two points correspond to the angles $0\,(0^\circ)$ and $\pi\,(180^\circ)$ or, more generally, to any angle that is an integral multiple of $\pi\,(180^\circ)$, such as $0\,(0^\circ)$, $\pm\pi\,(180^\circ)$, $\pm2\pi\,(360^\circ)$, $\pm3\pi\,(540^\circ)$, and so on. Such angles must be excluded from the domain of the cotangent and cosecant functions.

The domain of the cotangent function is the set of all real numbers, except integral multiples of $\pi\,(180^\circ)$.

The domain of the cosecant function is the set of all real numbers, except integral multiples of $\pi\,(180^\circ)$.

Next we determine the range of each of the six trigonometric functions. Refer again to Figure 1 above.
Let $P = (x, y)$ be the point on the unit circle that corresponds to the angle θ. It follows that $-1 \le x \le 1$ and $-1 \le y \le 1$. Since $\sin\theta = y$ and $\cos\theta = x$, it follows that $-1 \le \sin\theta \le 1$ and $-1 \le \cos\theta \le 1$.

The range of both the sine function and the cosine function consists of all real numbers between -1 and 1, inclusive. Using absolute value notation, we have $|\sin\theta| \le 1$ and $|\cos\theta| \le 1$.

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If θ is not an integral multiple of π (180°), then $\csc\theta = \dfrac{1}{y}$. Since $y = \sin\theta$ and $|y| = |\sin\theta| \le 1$, it follows that

$$|\csc\theta| = \frac{1}{|\sin\theta|} = \frac{1}{|y|} \ge 1$$

Since $\csc\theta = \dfrac{1}{y}$, the range of the cosecant function consists of all real numbers less than or equal to -1 or greater than or equal to 1.
That is, $\csc\theta \le -1$ or $\csc\theta \ge 1$.

Similarly, if θ is not an odd integral multiple of $\dfrac{\pi}{2}$ (90°), you can show the range of the secant function consists of all real numbers less than or equal to -1 or greater than or equal to 1. That is, $\sec\theta \le -1$ or $\sec\theta \ge 1$.

The range of both the tangent function and the cotangent function is the set of all real numbers.

$$-\infty < \tan\theta < \infty \qquad -\infty < \cot\theta < \infty$$

Summary:

Function	Symbol	Domain	Range
sine	$f(\theta) = \sin\theta$	All real numbers	All real numbers from -1 to 1, inclusive
cosine	$f(\theta) = \cos\theta$	All real numbers	All real numbers from -1 to 1, inclusive
tangent	$f(\theta) = \tan\theta$	All real numbers, except odd integral multiples of $\frac{\pi}{2}$ (90°)	All real numbers
cosecant	$f(\theta) = \csc\theta$	All real numbers, except integral multiples of π (180°)	All real numbers greater than or equal to 1 or less than or equal to -1
secant	$f(\theta) = \sec\theta$	All real numbers, except odd integral multiples of $\frac{\pi}{2}$ (90°)	All real numbers greater than or equal to 1 or less than or equal to -1
cotangent	$f(\theta) = \cot\theta$	All real numbers, except integral multiples of π (180°)	All real numbers

Determine the Period of the Trigonometric Functions

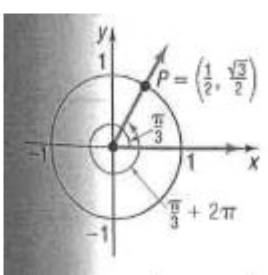

Figure 36 $\sin\left(\frac{\pi}{3} + 2\pi\right) = \sin\frac{\pi}{3}$; $\cos\left(\frac{\pi}{3} + 2\pi\right) = \cos\frac{\pi}{3}$

Figure 36 shows that for an angle of $\dfrac{\pi}{3}$ radians the corresponding point P on the unit circle is $\left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$. Notice that, for an angle of $\dfrac{\pi}{3} + 2\pi$ radians, the corresponding point P on the unit circle is also $\left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$.

Then $\sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$ and $\sin\left(\dfrac{\pi}{3} + 2\pi\right) = \dfrac{\sqrt{3}}{2}$ and

$\cos\dfrac{\pi}{3} = \dfrac{1}{2}$ and $\cos\left(\dfrac{\pi}{3} + 2\pi\right) = \dfrac{1}{2}$

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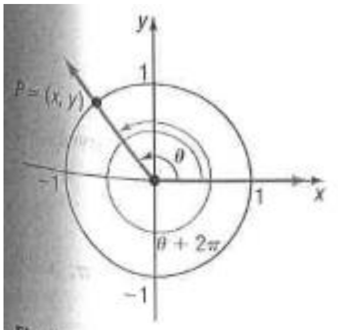

Figure 37 $\sin(\theta + 2\pi k) = \sin\theta$; $\cos(\theta + 2\pi k) = \cos\theta$

The example above illustrates a more general situation. For a given angle θ, measured in radians, suppose that you know the corresponding point $P = (x, y)$ on the unit circle. If you add 2π to θ, the point on the unit circle corresponding to $\theta + 2\pi$ is identical to the point P on the unit circle corresponding to θ. See Figure 37. The values of the trigonometric functions of $\theta + 2\pi$ are equal to the values of the corresponding trigonometric functions of θ.

If you add (or subtract) integral multiples of 2π to θ, the values of the sine and cosine function remain unchanged. That is, for all θ,

$\sin(\theta + 2\pi k) = \sin\theta$ and $\cos(\theta + 2\pi k) = \cos\theta$, where k is any integer.

Functions with this kind of behavior are called **periodic functions**.

A function f is called **periodic** if there is a positive number p such that, whenever θ is in the domain of f, so is $\theta + p$, and $f(\theta + p) = f(\theta)$.

If there is a smallest such number p, this smallest value is called the **(fundamental) period** of f.

Periodic Properties:
For k any integer,

$\sin\theta = \sin(\theta + 2\pi k)$	$\cos\theta = \cos(\theta + 2\pi k)$	$\tan\theta = \tan(\theta + \pi k)$
$\csc\theta = \csc(\theta + 2\pi k)$	$\sec\theta = \sec(\theta + 2\pi k)$	$\cot\theta = \cot(\theta + \pi k)$

So, sine, cosine, secant, and cosecant functions have period 2π. Once you know each of these function's values for $0 \le \theta < 2\pi$, you know all their values.

Tangent and cotangent functions have period π. Once you know each of these function's values for $0 \le \theta < \pi$, you know all their values.

Example 1: Find the exact values of:

a) $\sin 405^\circ = \sin(45^\circ + 360^\circ)$

$= \sin 45^\circ$ by the periodic property

$= \dfrac{y}{r}$

$= \dfrac{\sqrt{2}/2}{1}$ $\quad 45^\circ : (x, y) = \left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$

$= \dfrac{\sqrt{2}}{2}$

b) $\tan\dfrac{19\pi}{6} = \tan\left(\dfrac{\pi}{6} + 3\pi\right)$

$= \tan\left(\dfrac{\pi}{6}\right)$ by the periodic property

$= \dfrac{y}{x}$

$= \dfrac{1/2}{\sqrt{3}/2}$ $\quad \dfrac{\pi}{6} : (x, y) = \left(\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right)$

$= \dfrac{1}{\sqrt{3}}$

$= \dfrac{1}{\sqrt{3}}\left(\dfrac{\sqrt{3}}{\sqrt{3}}\right)$

$= \dfrac{\sqrt{3}}{3}$

The periodic properties of the trigonometric functions will be very helpful to you when you study their graphs later in the chapter.

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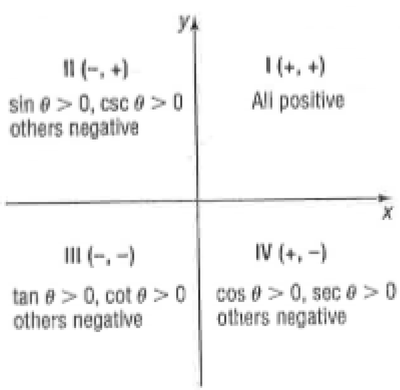
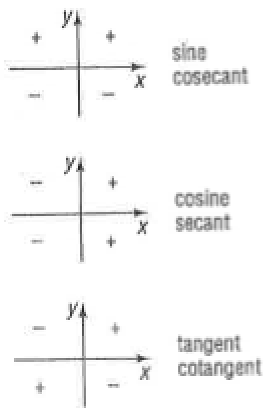
Determine the Signs of the Trigonometric Functions in a Given Quadrant

Let $P = (x, y)$ be the point on the unit circle that corresponds to angle θ. If you know in which quadrant the point P lies, then you can determine the signs of the trigonometric functions of θ.

Let θ be an angle in standard position, and let $P = (x, y)$ be a point on the terminal side of θ.

Quadrant of P	$\sin\theta$, $\csc\theta$	$\cos\theta$, $\sec\theta$	$\tan\theta$, $\cot\theta$
I	Positive	Positive	Positive
II	Positive	Negative	Negative
III	Negative	Negative	Positive
IV	Negative	Positive	Negative





Example 2: Name the quadrant in which the angle θ lies if $\sin\theta > 0$ and $\cos\theta < 0$.

Method 1:

Let $P = (x, y)$ be the point on the unit circle corresponding to θ.

Then $\sin\theta > 0$ and $\cos\theta < 0$

$\Rightarrow y > 0$ and $x < 0$

Thus, $(x, y) = (-, +)$.

$\Rightarrow \theta$ lies in quadrant II.

OR

Method 2:

$\sin\theta > 0$ for points P in quadrants I and II.

$\cos\theta < 0$ for points P in quadrants II and III.

Both are satisfied only if θ lies in quadrant II.