Section 6.3 – Properties of the Trigonometric Functions – Day 2

Find the Values of the Trigonometric Functions Using Fundamental Identities

If $P = (x, y)$ is the point on the unit circle corresponding to angle θ, then

$\sin\theta = y$ $\cos\theta = x$ $\tan\theta = \frac{y}{x},\ x \neq 0$

$\csc\theta = \frac{1}{y},\ y \neq 0$ $\sec\theta = \frac{1}{x},\ x \neq 0$ $\cot\theta = \frac{x}{y},\ y \neq 0$

Based on these definitions, we have the following:

Reciprocal Identities: $\csc\theta = \frac{1}{\sin\theta}$ $\sec\theta = \frac{1}{\cos\theta}$ $\cot\theta = \frac{1}{\tan\theta}$

Quotient Identities: $\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\cot\theta = \frac{\cos\theta}{\sin\theta}$

If $\sin\theta$ and $\cos\theta$ are known, then, using the reciprocal and quotient identities, it is easy to find the remaining trigonometric functions.

Example 3: Given $\sin\theta = \frac{2\sqrt{5}}{5}$ and $\cos\theta = \frac{\sqrt{5}}{5}$.
Find the exact value of each of the four remaining trigonometric functions of θ using identities.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{2\sqrt{5}/5}{\sqrt{5}/5} = \frac{2\sqrt{5}}{5}\left(\frac{5}{\sqrt{5}}\right) = 2$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{2\sqrt{5}/5} = \frac{5}{2\sqrt{5}} = \frac{5}{2\sqrt{5}}\left(\frac{\sqrt{5}}{\sqrt{5}}\right) = \frac{5\sqrt{5}}{2(5)} = \frac{\sqrt{5}}{2}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\sqrt{5}/5} = \frac{5}{\sqrt{5}} = \frac{5}{\sqrt{5}}\left(\frac{\sqrt{5}}{\sqrt{5}}\right) = \frac{5\sqrt{5}}{5} = \sqrt{5}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\sqrt{5}/5}{2\sqrt{5}/5} = \frac{\sqrt{5}}{5}\left(\frac{5}{2\sqrt{5}}\right) = \frac{1}{2}$$

(You could also find $\cot\theta$ by using the identity $\cot\theta = \frac{1}{\tan\theta}$, but if your tangent calculation is incorrect, then the cotangent will also be incorrect. So, to prevent the propagation of an error, it is best to use the given data.)

Pythagorean Identities:

$\sin^2\theta + \cos^2\theta = 1$ $\tan^2\theta + 1 = \sec^2\theta$ $1 + \cot^2\theta = \csc^2\theta$

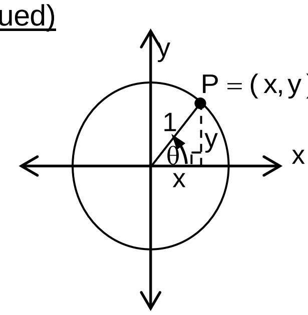
These identities can be obtained as follows:

If the point on the terminal side of angle θ is $P = (x, y)$ and if P lies on the unit circle, then $x^2 + y^2 = 1$.

But $x = \cos\theta$ and $y = \sin\theta$, so $\cos^2\theta + \sin^2\theta = 1$ or $\sin^2\theta + \cos^2\theta = 1$.

(It is customary to write $\sin^2\theta$ instead of $(\sin\theta)^2$.)

Section 6.3 – Properties of the Trigonometric Functions – Day 2 (continued)



If $\cos\theta \neq 0$, then dividing $\sin^2\theta + \cos^2\theta = 1$ by $\cos^2\theta$ gives

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\left(\frac{\sin\theta}{\cos\theta}\right)^2 + 1 = \left(\frac{1}{\cos\theta}\right)^2$$

$$\tan^2\theta + 1 = \sec^2\theta$$

Similarly, if $\sin\theta \neq 0$ and you divide by $\sin^2\theta$, you get $1 + \cot^2\theta = \csc^2\theta$.

Find the Exact Values of the Trigonometric Functions of an Angle Given One of the Functions and the Quadrant of the Angle

Many problems require finding the exact values of the remaining trigonometric functions when the value of one of them is known and the quadrant in which θ lies can be found. There are two approaches to solving such problems. One approach uses a circle of radius *r* ; the other uses identities.

When using identities, sometimes a rearrangement is required. For example, $\sin^2\theta + \cos^2\theta = 1$ can be solved for $\cos\theta$ in terms of $\sin\theta$ (or vice versa) as follows:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \sin^2\theta$$

$\cos\theta = \pm\sqrt{1-\sin^2\theta}$ where the + sign is used if $\cos\theta > 0$ and the – sign is used if $\cos\theta < 0$.

Using the other two Pythagorean Identities ($\tan^2\theta + 1 = \sec^2\theta$ and $1 + \cot^2\theta = \csc^2\theta$), you can similarly solve for $\tan\theta$, $\sec\theta$, $\cot\theta$, or $\csc\theta$.

Example 4: Use two methods to find the exact values of the remaining trigonometric functions given one trigonometric function and information about θ's location.

Given: $\sin\theta = \frac{12}{13}$ and θ is in quadrant II.

Method 1: Let $P = (x, y)$ be a point on the terminal side of θ.

Then $\sin\theta = \frac{y}{r}$ and $x^2 + y^2 = r^2$.

$$\sin\theta = \frac{12}{13}$$

$$= \frac{y}{r} \Rightarrow \text{Assume } y = 12,\ r = 13$$

$$x^2 + y^2 = r^2$$

$$x^2 = r^2 - y^2$$

$$= 13^2 - 12^2$$

$$= 169 - 144$$

$$= 25$$

$$x = \pm\sqrt{25}$$

$$= \pm 5$$

θ in quadrant II ⇒ $x = -5$, since x is negative in quadrant II

$\cos\theta = \frac{x}{r} = \frac{-5}{13}$ $\sec\theta = \frac{r}{x} = \frac{13}{-5}$ $\csc\theta = \frac{r}{y} = \frac{13}{12}$ $\tan\theta = \frac{y}{x} = \frac{12}{-5}$ $\cot\theta = \frac{x}{y} = \frac{-5}{12}$

Section 6.3 – Properties of the Trigonometric Functions – Day 2 (continued)

Example 4: (continued)

Method 2: $\sin^2\theta + \cos^2\theta = 1$

$\cos^2\theta = 1 - \sin^2\theta$

$\cos\theta = \pm\sqrt{1-\sin^2\theta}$

θ in quadrant II $\Rightarrow \cos\theta = -\sqrt{1-\sin^2\theta}$ cosine negative in quadrant II

$$\cos\theta = -\sqrt{1-\left(\frac{12}{13}\right)^2}$$
$$= -\sqrt{1-\frac{144}{169}}$$
$$= -\sqrt{\frac{169-144}{169}}$$
$$= -\sqrt{\frac{25}{169}}$$
$$= \frac{-5}{13}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{12}{13}}{\frac{-5}{13}} = \frac{-12}{5}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\frac{-5}{13}}{\frac{12}{13}} = \frac{-5}{12}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{-5}{13}} = \frac{-13}{5}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{12}{13}} = \frac{13}{12}$$

Summary: Finding the Values of the Trigonometric Functions of θ When the Value of One Function is Known and the Quadrant of θ is Known

Given the value of one trigonometric function and the quadrant in which θ lies, the exact value of each of the remaining five trigonometric functions can be found in either of two ways.

Option 1: Using a Circle of Radius r

Step 1: Draw a circle centered at the origin showing the location of the angle θ and the point $P = (x, y)$ that corresponds to θ. The radius of the circle that contains $P = (x, y)$ is $r = \sqrt{x^2 + y^2}$.

Step 2: Assign a value to two of the three variables x, y, r based on the value of the given trigonometric function and the location of P.

Step 3: Use the fact that P lies on the circle $x^2 + y^2 = r^2$ to find the missing variable.

Step 4: Find the remaining trigonometric functions using $\sin\theta = \frac{y}{r}, \cos\theta = \frac{x}{r}, \tan\theta = \frac{x}{y},$ etc.

Option 2: Identities

Use appropriately selected identities to find the value of each remaining trigonometric function.

Section 6.3 – Properties of the Trigonometric Functions – Day 2 (continued)

Use Even – Odd Properties to Find the Exact Values of the Trigonometric Functions

Recall that a function is even if $f(-x) = f(x)$ for all x in the domain of f. Even functions are symmetric with respect to the y-axis $\left((x, y) \Rightarrow (-x, y)\right)$. A function is odd if $f(-x) = -f(x)$ for all x in the domain of f. Odd functions are symmetric with respect to the origin $\left((x, y) \Rightarrow (-x, -y)\right)$.

The trigonometric functions sine, cosecant, tangent, and cotangent are odd functions.
The functions cosine and secant are even functions.

Theorem: Even-Odd Properties

$\sin(-\theta) = -\sin\theta$	$\cos(-\theta) = \cos\theta$	$\tan(-\theta) = -\tan\theta$
$\csc(-\theta) = -\csc\theta$	$\sec(-\theta) = \sec\theta$	$\cot(-\theta) = -\cot\theta$

Example 5: Find the exact value of:

a) $\sin(-60^\circ)$

$\sin(-60^\circ) = -\sin 60^\circ$ since sine is odd

$$= -\frac{y}{r}$$

$$= -\frac{\sqrt{3}/2}{1} \qquad 60^\circ : (x, y) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

$$= \frac{-\sqrt{3}}{2}$$

b) $\sec(-30^\circ)$

$\sec(-30^\circ) = \sec 30^\circ$ since secant is even

$$= \frac{r}{x}$$

$$= \frac{1}{\sqrt{3}/2} \qquad 30^\circ : (x, y) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

$$= \frac{2}{\sqrt{3}}$$

$$= \frac{2}{\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3}}\right)$$

$$= \frac{2\sqrt{3}}{3}$$

All material has been taken from *Precalculus,* by M. Sullivan, 10th Edition