Section 6.6 – Phase Shift

We have seen that the graph of $y = A\sin(\omega x)$, $\omega > 0$, has amplitude $|A|$ and period $T = \frac{2\pi}{\omega}$. One cycle can be drawn as x varies from 0 to $T = \frac{2\pi}{\omega}$, or equivalently, as ωx varies from 0 to 2π. See Figure 70 below.

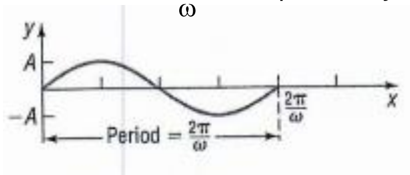


Figure 70 One cycle of $y = A\sin(\omega x), A > 0, \omega > 0$

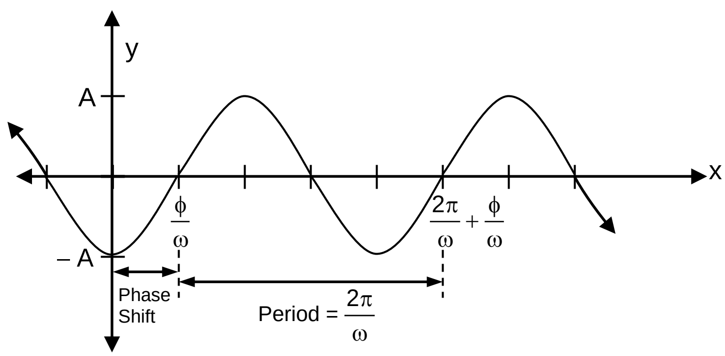
Now we want to consider the graph of $y = A\sin(\omega x - \phi)$

$$= A\sin\left[\omega\left(x - \frac{\phi}{\omega}\right)\right],$$ where $\omega > 0$ and ϕ are real numbers.

The graph is a sine curve with amplitude $|A|$. As $\omega x - \phi$ varies from 0 to 2π, one period will be traced out. This period begins when $\omega x - \phi = 0$ or $x = \frac{\phi}{\omega}$ and ends when $\omega x - \varphi = 2\pi$ or $x = \frac{\phi}{\omega} + \frac{2\pi}{\omega}$.

The beginning and end can also be found by solving the inequality:

$$0 \le \omega x - \phi \le 2\pi$$
$$\phi \le \omega x \le 2\pi + \phi$$
$$\frac{\phi}{\omega} \le x \le \frac{2\pi}{\omega} + \frac{\phi}{\omega}$$



Notice that the graph of $y = A\sin(\omega x - \phi) = A\sin\left[\omega\left(x - \frac{\phi}{\omega}\right)\right]$ is the same as the graph of $y = A\sin(\omega x)$, except that it has been shifted $\left|\frac{\phi}{\omega}\right|$ units (to the right if $\phi > 0$ and to the left if $\phi < 0$). This number $\frac{\phi}{\omega}$ is called the phase shift of the graph of $y = A\sin(\omega x - \phi)$.

For the graphs of $y = A\sin(\omega x - \phi) = A\sin\left[\omega\left(x - \frac{\phi}{\omega}\right)\right]$ or $y = A\cos(\omega x - \phi) = A\cos\left[\omega\left(x - \frac{\phi}{\omega}\right)\right]$, for $\omega > 0$,

Amplitude = $|A|$, Period = $T = \frac{2\pi}{\omega}$, and Phase Shift $= \frac{\phi}{\omega}$.

The phase shift is to the left if $\phi < 0$ and to the right if $\phi > 0$.

Section 6.6 – Phase Shift (continued)

Example 1: Find the amplitude, period, and phase shift of $f(x) = 4\sin(2x - \pi)$, and graph the function.

Label the five (5) key points on each graph in Example 1.

$$f(x) = A\sin(\omega x - \phi)$$
$$= 4\sin(2x - \pi)$$
$$= 4\sin\left[2\left(x - \frac{\pi}{2}\right)\right]$$

$\Rightarrow$ $A = 4$, $\omega = 2$, $\phi = \pi$ A is positive, so this is a non-reflected sine function.

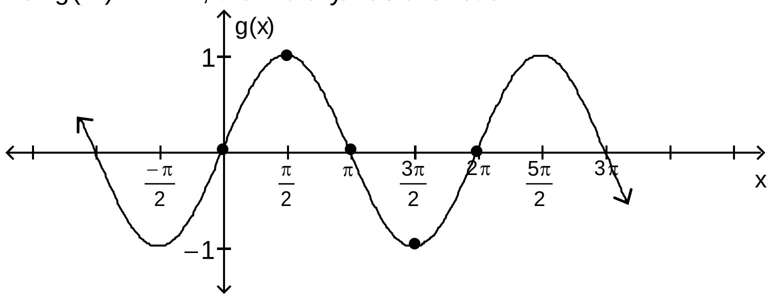
Amplitude $= |A| = |4| = 4$

Period $= T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{2} = \pi$

Phase Shift $= \dfrac{\phi}{\omega} = \dfrac{\pi}{2}$ $\phi > 0 \Rightarrow \dfrac{\pi}{2}$ units to the right

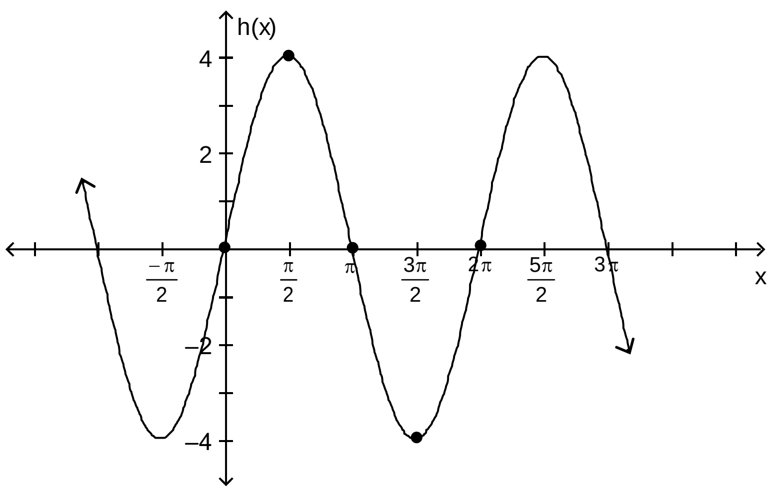
The graph is obtained by using transformations:

a) Let $g(x) = \sin x$, the Library/Basic function.



b) $h(x) = 4g(x)$

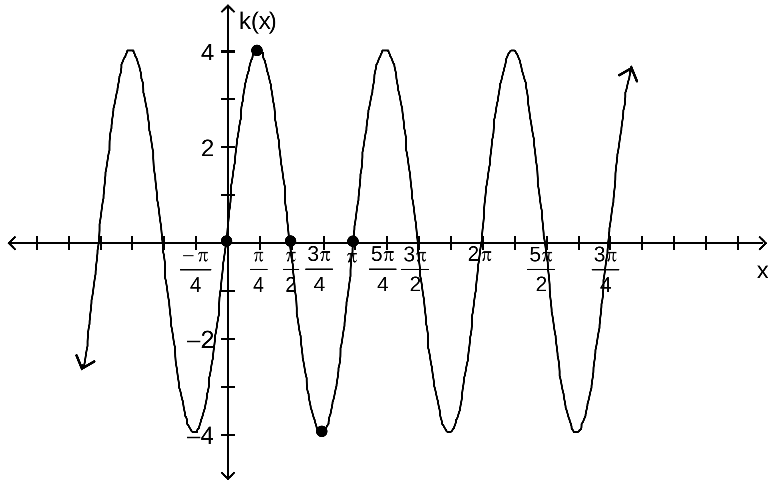
$= 4\sin x$, a vertical stretch by a factor of 4



Section 6.6 – Phase Shift (continued)

c) $k(x) = h(2x)$

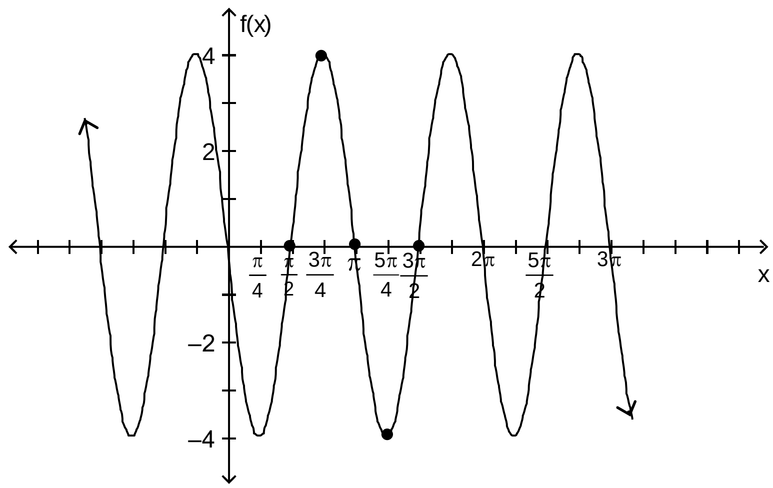
$= 4\sin(2x)$ a horizontal compression by a factor of $\frac{1}{2}$



d) $f(x) = k\left(x - \frac{\pi}{2}\right)$ a shift right $\frac{\pi}{2}$ units

$= 4\sin\left[2\left(x - \frac{\pi}{2}\right)\right]$

$= 4\sin(2x - \pi)$



You may also use the Five Key Points method to graph these functions.

Steps for Graphing Sinusoidal Functions $y = A\sin(\omega x - \phi) + B$ **or** $y = A\cos(\omega x - \phi) + B$

Step 1: Determine the amplitude $|A|$, period $T = \frac{2\pi}{\omega}$, and phase shift $\frac{\phi}{\omega}$.

Step 2: Determine the starting point of one cycle of the graph, $\frac{\phi}{\omega}$. Determine the ending point of one cycle of the graph, $\frac{\phi}{\omega} + \frac{2\pi}{\omega}$. Divide the interval $\left[\frac{\phi}{\omega}, \frac{\phi}{\omega} + \frac{2\pi}{\omega}\right]$ into four subintervals, each of length $\frac{2\pi}{\omega} \div 4$.

Step 3: Use the endpoints of the subintervals to find the five key points on the graph.

Step 4: Plot the five key points, and connect them with a sinusoidal graph to obtain one cycle of the graph. Extend the graph in each direction to make it complete.

Step 5: If $B \neq 0$, apply a vertical shift.

Section 6.6 – Phase Shift (continued)

So, repeating Example 1 using the Five Key Points method, you have

Step1: $f(x) = A\sin(\omega x - \phi)$

$= 4\sin(2x - \pi)$

$= 4\sin\left[2\left(x - \frac{\pi}{2}\right)\right]$

$\Rightarrow \quad A = 4,\ \omega = 2,\ \phi = \pi$ A is positive, so this is a non-reflected sine function.

Amplitude $= |A| = |4| = 4$ Period $= T = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi$ Phase Shift $= \frac{\phi}{\omega} = \frac{\pi}{2}$ $\phi > 0 \Rightarrow \frac{\pi}{2}$ units to the right

Step2: The graph of $f(x) = A\sin(\omega x - \phi)$ will lie between -4 and 4 on the y-axis. One cycle will begin at $x = \frac{\phi}{\omega} = \frac{\pi}{2}$ and end at $x = \frac{\phi}{\omega} + \frac{2\pi}{\omega} = \frac{\pi}{2} + \frac{2\pi}{2} = \frac{3\pi}{2}$

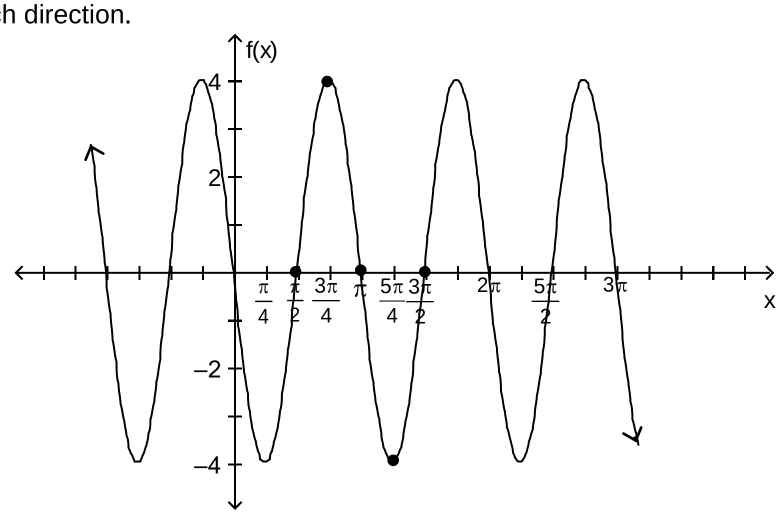
To find the five key points, divide the interval $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ into four subintervals, each of length $\pi \div 4 = \frac{\pi}{4}$, by finding the following values of x:

1st x-coordinate	2nd x-coordinate	3rd x-coordinate	4th x-coordinate	5th x-coordinate
$\frac{\pi}{2}$	$\frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$	$\frac{3\pi}{4} + \frac{\pi}{4} = \pi$	$\pi + \frac{\pi}{4} = \frac{5\pi}{4}$	$\frac{5\pi}{4} + \frac{\pi}{4} = \frac{6\pi}{4} = \frac{3\pi}{2}$

Step3: Use these values to determine the five key points on the graph:

$\left(\frac{\pi}{2}, 0\right),\ \left(\frac{3\pi}{4}, 4\right),\ (\pi, 0),\ \left(\frac{5\pi}{4}, -4\right),\ \left(\frac{3\pi}{2}, 0\right)$

Step 4: Plot these points and fill in the graph of the sine function as shown below. Extend the graph in each direction.



All material has been taken from *Precalculus,* by M. Sullivan, 10th Edition