Section 7.1 – The Inverse Trigonometric Functions

In Section 5.2 we discussed inverse functions, and we concluded that if a function is one-to-one, it will have an inverse function. Also, if a function is not one-to-one, it may be possible to restrict its domain in some suitable manner so that the restricted function is one-to-one. For example, the function $y = x^2$ is not one-to-one; however, if the domain is restricted to $x \ge 0$, the function is one-to-one.

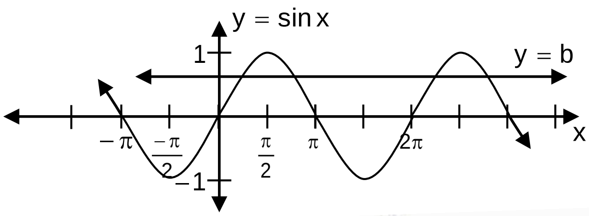
From Section 5.2: A function f is said to be one-to-one if, for any choice of numbers x_1 and x_2 in the domain of f, with $x_1 \ne x_2$, then $f(x_1) \ne f(x_2)$.

Horizontal Line Test: If every horizontal line intersects the graph of a function f in at most one point, then f is one-to-one.

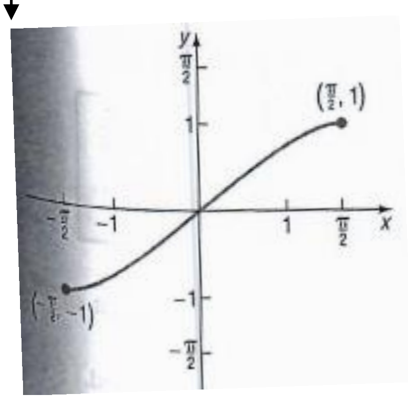
Other properties of a one-to-one function f and its inverse function f^{-1} that were discussed in Section 5.2 are:

1) $f^{-1}(f(x)) = x$ for x in the domain of f, and $f\left(f^{-1}(x)\right) = x$ for every x in the domain of f^{-1}.
2) The domain of f = range of f^{-1}, and the range of f = domain of f^{-1}.
3) The graph of f and f^{-1} are reflections of one another about the line $y = x$.
4) If a function $y = f(x)$ has an inverse function, the implicit equation of the inverse function is $x = f(y)$. If you solve this equation for y, you obtain the explicit equation $y = f^{-1}(x)$.

The Inverse Sine Function



Every horizontal line $y = b$, where $-1 \le b \le 1$, intersects the graph of $y = \sin x$ infinitely many times. $\Rightarrow$ It fails the horizontal line test. $\Rightarrow$ It is not a one-to-one function.



But, if we restrict the domain of $y = \sin x$ to the interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, the restricted function $y = \sin x$, $\frac{-\pi}{2} \le x \le \frac{\pi}{2}$ is one-to-one and will have an inverse function.

From section 5.2, you know you have to interchange x and y to find the inverse:

$y = \sin x$

$x = \sin y$, $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$ Implicit form

$y = \sin^{-1} x$, $-1 \le x \le 1$ Explicit form, called the **inverse sine of x**. Symbolized by $y = f^{-1}(x) = \sin^{-1} x$

So, $y = \sin^{-1} x$ means $x = \sin y$ where $-1 \le x \le 1$ and $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$.

$y = \sin^{-1} x$ is read as "y is the angle or real number whose sine equals x" or "y is the inverse sine of x."

Be careful with the notation used. The superscript -1 that appears in $y = \sin^{-1} x$ is not an exponent but the symbol used to denote the inverse function f^{-1} of f. You will sometimes see $y = \sin^{-1} x$ written as $y = \arcsin x$.

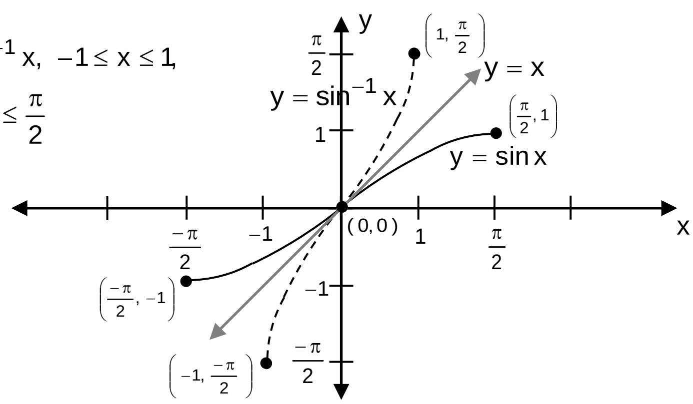
Section 7.1 – The Inverse Trigonometric Functions (continued)

The inverse of a function f receives as input an element from the range of f and returns as output an element in the domain of f. The restricted sine function, $y = f(x) = \sin x$, receives as input an angle or real number x in the interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ and outputs a real number in the interval $[-1, 1]$. Therefore, the inverse sine function $y = \sin^{-1} x$, receives as input a real number in the interval $[-1, 1]$ or $-1 \le x \le 1$, its domain, and outputs an angle or real number in the interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ or $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$, its range.

The graph of the inverse sine function, $y = \sin^{-1} x$, can be obtained by reflecting the restricted portion of the graph of $y = f(x) = \sin x$ about the line $y = x$, as shown in the graph below.

$y = \sin^{-1} x, \ -1 \le x \le 1,$

$\frac{-\pi}{2} \le y \le \frac{\pi}{2}$



Example 1: Find the exact value of $\sin^{-1}(-1)$.

Let $\theta = \sin^{-1}(-1)$, $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$.

$\Rightarrow \sin\theta = -1$, $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$

Now, $\sin\left(\frac{-\pi}{2}\right) = -1$ and $\frac{-\pi}{2}$ lies in $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

Thus, $\theta = \sin^{-1}(-1)$

$= \frac{-\pi}{2}$

Example 2: Find the exact value of $\sin^{-1}\left(\frac{1}{2}\right)$.

Let $\theta = \sin^{-1}\left(\frac{1}{2}\right)$, $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$.

$\Rightarrow \sin\theta = \frac{1}{2}$, $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$.

Now, $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ and $\frac{\pi}{6}$ lies in $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

Thus, $\theta = \sin^{-1}\left(\frac{1}{2}\right)$

$= \frac{\pi}{6}$

For most numbers x, the value $y = \sin^{-1} x$ must be approximated. Use your calculator to approximate the value.

Example 3: Find $\sin^{-1}(-0.46)$ in radians rounded to two decimal places.

$\sin^{-1}(-0.46) \approx -0.4779$

≈ -0.48 Check: $\sin(-0.48) \approx -0.461$

Use Properties of Inverse Functions to Find Exact Values of Certain Composite Functions

Recall, from section 5.2, that $f^{-1}(f(x)) = x$ for all x in the domain of f and that $f\left(f^{-1}(x)\right) = x$ for all x in the domain of f^{-1}. In terms of the sine function and its inverse, these properties are of the form

$\sin^{-1}(\sin x) = x$, where $\frac{-\pi}{2} \le x \le \frac{\pi}{2}$ (2A)

$\sin\left(\sin^{-1} x\right) = x$, where $-1 \le x \le 1$ (2B)

Section 7.1 – The Inverse Trigonometric Functions (continued)

Example 4: Find the exact value of a) $\sin^{-1}\left(\sin\frac{3\pi}{8}\right)$ and b) $\sin^{-1}\left(\sin\frac{3\pi}{4}\right)$.

a) $\sin^{-1}\left(\sin\frac{3\pi}{8}\right)$ Because $\frac{3\pi}{8}$ is in the interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, by property 2A above,

$$\sin^{-1}\left(\sin\frac{3\pi}{8}\right) = \frac{3\pi}{8}.$$

b) $\sin^{-1}\left(\sin\frac{3\pi}{4}\right)$ Because $\frac{3\pi}{4}$ is not in the interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, first find an angle θ in the interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ for which $\sin\theta = \sin\frac{3\pi}{4}$. Now, $\sin\frac{3\pi}{4} = \sin\frac{\pi}{4}$ and $\frac{\pi}{4}$ is in interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, so by property 2A above, $\sin^{-1}\left(\sin\frac{3\pi}{4}\right) = \sin^{-1}\left(\sin\frac{\pi}{4}\right)$

$$= \frac{\pi}{4}$$

The Inverse Cosine Function

For the inverse cosine function, restrict the domain of $y = \cos x$ to the interval $[0, \pi]$.

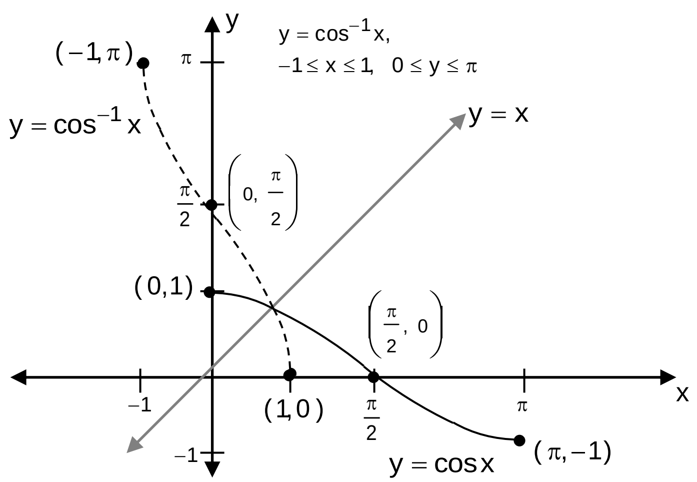
The inverse cosine of x:

$y = \cos^{-1} x$ means $x = \cos y$

where $-1 \le x \le 1$ and $0 \le y \le \pi$

You have $\cos^{-1}(\cos x) = x$, where $0 \le x \le \pi$

$\cos\left(\cos^{-1} x\right) = x$, where $-1 \le x \le 1$



The Inverse Tangent Function

For the inverse tangent function, restrict the domain of $y = \tan x$ to the interval $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

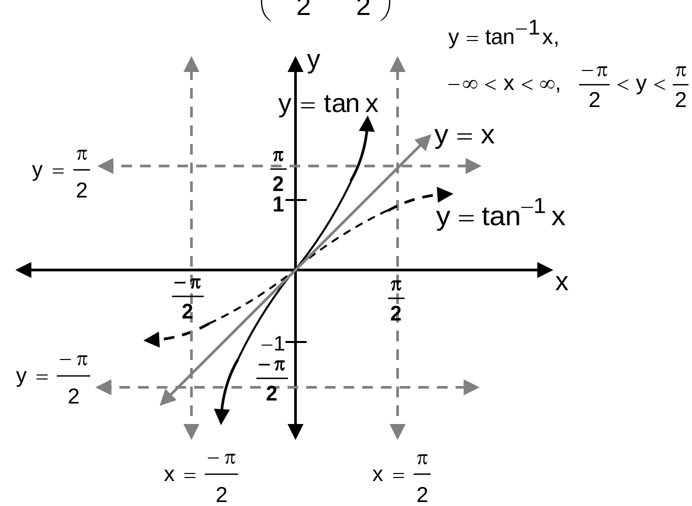
The inverse tangent of x:

$y = \tan^{-1} x$ means $x = \tan y$

where $-\infty < x < \infty$ and $\frac{-\pi}{2} < y < \frac{\pi}{2}$

You have $\tan^{-1}(\tan x) = x$, where $\frac{-\pi}{2} < x < \frac{\pi}{2}$

$\tan\left(\tan^{-1} x\right) = x$, where $-\infty < x < \infty$



Section 7.1 – The Inverse Trigonometric Functions (continued)

Example 5: Find the exact value of $\tan^{-1}\left(-\sqrt{3}\right)$.

Let $\theta = \tan^{-1}\left(-\sqrt{3}\right),\ \frac{-\pi}{2} < \theta < \frac{\pi}{2}$.

$\Rightarrow \tan\theta = -\sqrt{3},\ \frac{-\pi}{2} < \theta < \frac{\pi}{2}$

Now, $\tan\left(\frac{-\pi}{3}\right) = -\sqrt{3}$ and $\frac{-\pi}{3}$ lies in $\left(\frac{-\pi}{2},\ \frac{\pi}{2}\right)$.

Thus, $\theta = \tan^{-1}\left(-\sqrt{3}\right)$

$= \frac{-\pi}{3}$

We will solve the following compound types of problems two ways.

Method 1 is to find angle θ when solving.

Example 6: - Method 1

Find the exact value of $\cos\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$.

Let $\theta = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right),\ \frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$.

$\Rightarrow \sin\theta = \frac{\sqrt{2}}{2},\ \frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$

Now, $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ and $\frac{\pi}{4}$ lies in $\left[\frac{-\pi}{2},\ \frac{\pi}{2}\right]$.

Thus, $\theta = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$

$= \frac{\pi}{4}$

So, $\cos\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right) = \cos\theta$

$= \cos\left(\frac{\pi}{4}\right)$

$= \frac{x}{r} \qquad \frac{\pi}{4} : \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$= \frac{\frac{\sqrt{2}}{2}}{1}$

$= \frac{\sqrt{2}}{2}$

Example 7: - Method 1

Find the exact value of $\sec\left(\cos^{-1}\left(\frac{1}{2}\right)\right)$.

Let $\theta = \cos^{-1}\left(\frac{1}{2}\right),\ 0 \le \theta \le \pi$.

$\Rightarrow \cos\theta = \frac{1}{2},\ 0 \le \theta \le \pi$

Now, $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ and $\frac{\pi}{3}$ lies in $[0,\ \pi]$.

Thus, $\theta = \cos^{-1}\left(\frac{1}{2}\right)$

$= \frac{\pi}{3}$

So, $\sec\left(\cos^{-1}\left(\frac{1}{2}\right)\right) = \sec\theta$

$= \sec\left(\frac{\pi}{3}\right)$

$= \frac{r}{x} \qquad \frac{\pi}{3} : \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$= \frac{1}{\left(\frac{1}{2}\right)}$

$= 2$